On stellated spheres, shellable balls, lower bounds and a combinatorial criterion for tightness

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Abstract

We introduce the k-stellated (combinatorial) spheres and compare and contrast them with k-stacked (triangulated) spheres. It is shown that for $d \geq 2k$, any k-stellated sphere of dimension d bounds a unique and canonically defined k-stacked ball. In parallel, any k-stacked polytopal sphere of dimension $d \geq 2k$ bounds a unique and canonically defined k-stacked (polytopal) ball, which answers a question of McMullen. We consider the class $\mathcal{W}_k(d)$ of combinatorial d-manifolds with k-stellated links. For $d \geq 2k + 2$, any member of $\mathcal{W}_k(d)$ bounds a unique and canonically defined "k-stacked" $(d + 1)$-manifold.

We introduce the mu-vector of simplicial complexes, and show that the mu-vector of any 2-neighbourly simplicial complex dominates its vector of Betti numbers componentwise, and the two vectors are equal precisely when the complex is tight. When $d \geq 2k$, we are able to estimate/compute certain alternating sums of the mu-numbers of any 2-neighbourly member of $\mathcal{W}_k(d)$. This leads to a lower bound theorem for such triangulated manifolds. As an application, it is shown that any $(k + 1)$-neighbourly member of $\mathcal{W}_k(d)$ is tight, subject only to an extra condition on the $k^{\rm th}$ Betti number in case $d = 2k + 1$. This result more or less settles a recent conjecture of Effenberger, and it also provides a uniform and conceptual tightness proof for all the known tight triangulated manifolds, with only two exceptions. It is shown that any polytopal upper bound sphere of odd dimension $2k + 1$ belongs to the class $\mathcal{W}_k(2k + 1)$, thus generalizing a theorem (the $k = 1$ case) due to Perles. This shows that the case $d = 2k + 1$ is indeed exceptional for the tightness theorem.

We also prove a lower bound theorem for triangulated manifolds in which the members of $\mathcal{W}_1(d)$ provide the equality case. This generalises a result (the $d = 4$ case) due to Walkup and Kühnel. As a consequence, it is shown that every tight member of $\mathcal{W}_1(d)$ is strongly minimal, thus providing substantial evidence in favour of a conjecture of Kühnel and Lutz.

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Contents

1 Summary of results

But for some exceptions in Section 3, all simplicial complexes considered here are finite and abstract. By a triangulated sphere/ball/manifold, we mean an abstract simplicial complex whose geometric carrier is a sphere/ball/manifold. We identify two complexes if they are isomorphic.

In this paper, we introduce the class $\Sigma_k(d)$, $0 \le k \le d+1$, of k-stellated triangulated d-spheres and compare it with the class $\mathcal{S}_k(d)$, $0 \le k \le d$, of k-stacked triangulated d-spheres. We have the filtration

$$\Sigma_0(d) \subseteq \Sigma_1(d) \subseteq \cdots \subseteq \Sigma_d(d) \subseteq \Sigma_{d+1}(d)$$

of the class of all combinatorial d-spheres, and the comparable filtration

$$\mathcal{S}_0(d) \subseteq \mathcal{S}_1(d) \subseteq \cdots \subseteq \mathcal{S}_d(d)$$

of the class of all triangulated d-spheres. The *standard d-sphere* S_{d+2}^d is the unique $(d+2)$-vertex triangulation of the d-sphere. It may be described as the boundary complex of the $(d+1)$-dimensional geometric simplex. The standard sphere S_{d+2}^d is the unique member of $\Sigma_0(d) = \mathcal{S}_0(d)$. We also have the equality $\Sigma_1(d) = \mathcal{S}_1(d)$. In the existing literature, the members of $\mathcal{S}_1(d)$ are known as the d-dimensional *stacked spheres*. For $d \ge 2k-1$, we have the inclusion $\Sigma_k(d) \subseteq \mathcal{S}_k(d)$. However, for each $k \ge 2$, there are k-stacked spheres which are not k-stellated.

In parallel with these classes of triangulated spheres, we also consider the classes $\widehat{\Sigma}_k(d)$ and $\widehat{\mathcal{S}}_k(d)$ of k-shelled d-balls and k-stacked d-balls, respectively. We have the filtration

$$\widehat{\Sigma}_0(d) \subseteq \widehat{\Sigma}_1(d) \subseteq \cdots \subseteq \widehat{\Sigma}_d(d)$$

of the class of all shellable d-balls, and the comparable filtration

$$\widehat{\mathcal{S}}_0(d) \subseteq \widehat{\mathcal{S}}_1(d) \subseteq \cdots \subseteq \widehat{\mathcal{S}}_d(d)$$

of the class of all triangulated d-balls. The *standard d-ball* B_{d+1}^d is the unique $(d+1)$-vertex triangulation of the d-dimensional ball. It may be described as the face complex

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of the d-dimensional geometric simplex. The standard ball B_{d+1}^d is the unique member of $\widehat{\Sigma}_0(d) = \widehat{\mathcal{S}}_0(d)$. We also have the equality $\widehat{\Sigma}_1(d) = \widehat{\mathcal{S}}_1(d)$ and for all $d \geq k$ we have the inclusion $\widehat{\Sigma}_k(d) \subseteq \widehat{\mathcal{S}}_k(d)$. However, for each $k \geq 2$, there are k-stacked balls which are not k-shelled.

While a k-*stellated* d-*sphere* is defined as a triangulated d-sphere which may be obtained from S_{d+2}^d by a finite sequence of bistellar moves of index $< k$, a k-*shelled* d-*ball* is a triangulated d-ball obtained from B_{d+1}^d by a finite sequence of shelling moves of index $< k$. A k-*stacked* d-*ball* is a triangulated d-ball all whose faces of codimension $k+1$ (i.e., dimension $d - k - 1$) are in its boundary. A k-*stacked* d-*sphere* is a triangulated d-sphere which may be represented as the boundary of a k-stacked $(d + 1)$-ball. The boundary of any k-shelled $(d + 1)$-ball is a k-stellated d-sphere. Conversely, when $d \geq 2k - 1$, any k-stellated d-sphere may be represented as the boundary of a k-shelled $(d + 1)$-ball. A triangulated ball is k-shelled if and only if it is k-stacked and shellable. Each k-stacked (respectively k-shelled) ball is the antistar of a vertex in a k-stacked (respectively k-stellated) sphere. We prove that, when $d \geq 2k$, for any k-stellated d-sphere S, there is a unique k-stacked $(d + 1)$-ball $\overline{S}$ whose boundary is S. The ball $\overline{S}$ has a natural and intrinsic description in terms of the combinatorics of S. We show that this result is also valid if $d \geq 2k$ and S is a polytopal k-stacked d-sphere, thus answering a query implicit in [33] raised by McMullen in the context of equality in GLBT (generalized lower bound theorem) for polytopal spheres. For general k-stacked spheres, we can only prove that, when $d \geq 2k + 1$, a k-stacked d-sphere S bounds a unique k-stacked ball $\overline{S}$. However, there seems to be no combinatorial description of $\overline{S}$ in this generality.

The entire g-vector (equivalently, f-vector) of a k-stellated d-sphere is determined by the k numbers $g_1, \ldots, g_k$. This is actually true, more generally, of k-stacked d-spheres. However, for a k-stellated sphere of dimension $d \geq 2k - 1$, these k components of its g-vector have an interesting geometric interpretation. For $1 \leq i \leq k$, g_i is the number of bistellar moves of index $i - 1$ in any sequence of bistellar moves (of index $\leq k - 1$) used to obtain the given d-sphere from the standard d-sphere.

Next, we introduce the class $\mathcal{W}_k(d)$, $0 \leq k \leq d$, of (combinatorial) d-manifolds whose vertex-links are k-stellated spheres. This class may be compared with the generalized Walkup classes $\mathcal{K}_k(d)$ of triangulated d-manifolds all whose vertex-links are k-stacked spheres. We have the inclusions $\Sigma_k(d) \subseteq \mathcal{W}_k(d)$, $\mathcal{S}_k(d) \subseteq \mathcal{K}_k(d)$ and, for $d \geq 2k$, $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$. In consequence, all the proper face-links (of dimension $\geq k - 1$) of members of $\mathcal{W}_k(d)$ are k-stellated, and all the proper face-links (of dimension $\geq k$) of members of $\mathcal{K}_k(d)$ are k-stacked. We have the filtration

$$\mathcal{W}_0(d) \subseteq \mathcal{W}_1(d) \subseteq \cdots \subseteq \mathcal{W}_d(d)$$

of the class of all closed combinatorial d-manifolds, and the corresponding filtration

$$\mathcal{K}_0(d) \subseteq \mathcal{K}_1(d) \subseteq \cdots \subseteq \mathcal{K}_{d-1}(d)$$

of the class of all triangulated closed d-manifolds all whose vertex-links are triangulated $(d - 1)$-spheres. Again, the standard sphere S_{d+2}^d is the only member of $\mathcal{W}_0(d) = \mathcal{K}_0(d)$, and we have $\mathcal{W}_1(d) = \mathcal{K}_1(d)$. For $d \geq 2k + 2$, any member M of $\mathcal{W}_k(d)$ is the boundary of a canonically defined $(d + 1)$-manifold $\overline{M}$ such that all the faces of codimension $k + 1$ in $\overline{M}$ belong to the boundary M.

The entire g-vector (equivalently, face-vector) of any member of $\mathcal{W}_k(d)$ is determined by the $(k+1)$ numbers $g_1, \ldots, g_{k+1}$. In particular, when $d \geq 2k$ is even, the Euler characteristic

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χ of any member of $\mathcal{W}_k(d)$ is determined by the $(k+1)^{\text{st}}$ component g_{k+1} of its g-vector by the formula

$$(-1)^k \binom{d+2}{k+1}(\chi - 2) = 2\, g_{k+1}.$$

If $M \in \mathcal{W}_k(d)$ is 2-neighbourly, with Betti numbers β_i (with respect to any field) and g-components g_i, then we show that

(a) $g_{l+1} \geq \binom{d+2}{l+1} \sum_{i=1}^{l} (-1)^{l-i}\beta_i$ for $1 \leq l < k$, when $d \geq 2k$,

(b) $g_{k+1} \geq \binom{d+2}{k+1} \sum_{i=1}^{k} (-1)^{k-i}\beta_i$, when $d = 2k+1$,

(c) $g_{k+1} = \binom{d+2}{k+1} \sum_{i=1}^{k} (-1)^{k-i}\beta_i$, when $d \geq 2k+2$, and

(d) $\beta_i = 0$ for $k+1 \leq i \leq d-k-1$, when $d \geq 2k+2$.

Since the components of the face vector are non-negative linear combinations of the g-numbers, this result may be interpreted as a lower bound theorem for 2-neighbourly members of $\mathcal{W}_k(d)$.

We also prove a lower bound theorem for general triangulated closed manifolds. When $d \geq 4$, among all triangulated closed d-manifolds with given first Betti number and given number of vertices, members of $\mathcal{W}_1(d)$ (when they exist) minimize the face vector componentwise. The case $d = 4$ of this result is due to Walkup and Kühnel.

Any member of $\Sigma_k(d)$ (or even of $\mathcal{S}_k(d)$) is at most k-neighbourly, unless it is the standard sphere. In consequence, any member of $\mathcal{W}_k(d)$ (or $\mathcal{K}_k(d)$), other than the standard sphere, is at most $(k+1)$-neighbourly. This leads us to consider the class $\mathcal{W}_k^*(d)$ (and $\mathcal{K}_k^*(d)$) consisting of all $(k+1)$-neighbourly members of $\mathcal{W}_k(d)$ (respectively of $\mathcal{K}_k(d)$). We have $\mathcal{W}_k^*(d) \subseteq \mathcal{K}_k^*(d)$ for all k and d. These classes have no member other than the standard spheres unless $d \geq 2k$. Generalizing a result of Perles, we prove that any upper bound polytopal sphere (for instance, cyclic sphere) of odd dimension $2k+1$ belongs to the class $\mathcal{W}_k^*(2k+1)$.

Moreover, when $d \geq 2k+2$ and $k \geq 2$, any member of $\mathcal{W}_k^*(d)$ has the same integral homology as the connected sum of β copies of $S^k \times S^{d-k}$, where the non-negative integer β is given by the formula

$$\binom{m+k-d-2}{k+1} = \binom{d+2}{k+1}\beta,$$

with $m = f_0$, the number of vertices. This result may be compared with Kalai's theorem: for $d \geq 4$, any member of $\mathcal{W}_1(d)$ triangulates the connected sum of finitely many copies of $S^1 \times S^{d-1}$ or $S^{d-1} \ltimes S^1$.

Recall that a connected simplicial complex X is said to be *tight* with respect to a field $\mathbb{F}$ if the inclusion map from any induced (full) subcomplex of X into X is injective at the level of $\mathbb{F}$-homology. In case of a closed manifold X, this has the following geometric interpretation: X is $\mathbb{F}$-tight if the standard geometric realization of X in $\mathbb{R}^{n-1}$ (n = number of vertices of X) is "as convex as possible" subject to the constraint imposed by its homology with $\mathbb{F}$-coefficients. Our interest in the notion of tightness stems from the following conjecture of Kühnel and Lutz (which seems to be borne out by all the known examples): any tight

triangulated manifold has the componentwise minimum face vector among all triangulations of the same manifold! As a consequence of our lower bound theorem for general triangulated manifolds, we show that this conjecture is valid for all tight members of $\mathcal{W}_1(d)$. Since any connected induced subcomplex of a tight simplicial complex is obviously tight, tightness imposes an extremely powerful constraint on the possible combinatorics of a simplicial complex. For instance, any $\mathbb{F}$-tight simplicial complex is necessarily 2-neighbourly and any $\mathbb{F}$-tight triangulated closed manifold is $\mathbb{F}$-orientable. Thus, it is not surprising that, apart from three infinite families (including the trivial family of standard spheres), only seventeen sporadic examples of tight triangulated manifolds (of dimension > 2) are known so far.

All this makes it very important to obtain usable combinatorial criteria for tightness of triangulated manifolds. In this paper, we introduce the mu-vector of a simplicial complex (with respect to a given field) and compare it with its beta-vector (i.e., the vector of Betti numbers over the same field). It is shown that, in general, for 2-neighbourly simplicial complexes, the alternating sums of the components of the mu-vector dominate the corresponding sums for the beta-vector; the two vectors coincide precisely in the case of tight complexes. This paraphrases the "combinatorial strong Morse inequality". We succeed in explicitly computing or estimating certain functionals of the mu-vectors of 2-neighbourly members of $\mathcal{W}_k(d)$, $d \geq 2k$, entirely in terms of their g-vectors. The lower bound theorem for $\mathcal{W}_k(d)$ stated above (as well as the determination of the homology type of members of $\mathcal{W}_k^*(d)$) is a consequence of this theory. It also leads to the following combinatorial criterion for tightness: for $d \neq 2k + 1$, any member M of $\mathcal{W}_k^*(d)$ is tight. This is with respect to any field in case $k \geq 2$, and with respect to a field $\mathbb{F}$ for which M is $\mathbb{F}$-orientable in case $k = 1$. (The example of upper bound polytopal spheres of odd dimension shows that the case $d = 2k + 1$ is a genuine exception to this result. We also find a characterization of the tight members of $\mathcal{W}_k^*(2k + 1)$, thus covering this exceptional case.) The $k = 1$ case of this theorem is a recent result due to Effenberger. Effenberger also conjectured the tightness of all members of the supposedly larger class $\mathcal{K}_k^*(d)$. Thus this paper, which is largely motivated by Effenberger's work, partially settles his conjecture. It may also be pointed out that we do not know of a single member of $\mathcal{K}_k^*(d)$ which is not in $\mathcal{W}_k^*(d)$.

In the final section of this paper, we present various examples, counter examples, questions and conjectures related to the above results. For instance, we show that for each $k \geq 2$, there are k-stacked triangulated d-spheres which are not even $(d + 1)$-stellated (i.e., not combinatorial spheres) and k-stacked combinatorial d-spheres which are not d-stellated. Recently, Klee and Novik found an extremely beautiful construction of a $(2d + 4)$-vertex triangulation M of $S^k \times S^{d-k}$ for all pairs $0 \leq k \leq d$. We show that, for $d \geq 2k$, these triangulations are in $\mathcal{W}_k(d)$. Klee and Novik obtained their triangulation M as the boundary complex of a triangulated $(d + 1)$-manifold $\overline{M}$. For $d \geq 2k + 2$, this is an instance of our canonical construction $M \mapsto \overline{M}$. As an application, we show that, for $d \neq 2k$, the full automorphism group of the Klee-Novik triangulation is a group of order $4d + 8$, already found by these authors. This makes it interesting to determine the full automorphism group of the Klee-Novik manifolds for $d = 2k$.

We show that the tightness of most of the known tight manifolds follows from our result. This provides a unified and conceptual proof of tightness of these manifolds, where the previous proofs were mostly by computer-aided case by case analysis.

In view of our (rather isolated) result on polytopal spheres, it seems natural to conjecture that for polytopal spheres of dimension $d \geq 2k$, the notions "k-stellated" and "k-stacked" coincide. We also pose a general lower bound conjecture to which members of $\mathcal{W}_k(d)$ (or $\mathcal{K}_k(d)$) should provide the cases of equality. This is related to a recent work of Novik and

Swartz, who proved a previous conjecture of Kalai.

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2 Bistellar moves and shelling moves

A d-dimensional simplicial complex is called *pure* if all its maximal faces (called *facets*) are d-dimensional. A d-dimensional pure simplicial complex is said to be a *weak pseudomanifold* if each of its $(d-1)$-faces is in at most two facets. For a d-dimensional weak pseudomanifold X, the *boundary* ∂X of X is the pure subcomplex of X whose facets are those $(d-1)$-dimensional faces of X which are contained in unique facets of X. The *dual graph* $\Lambda(X)$ of a weak pseudomanifold X is the graph whose vertices are the facets of X, where two facets are adjacent in $\Lambda(X)$ if they intersect in a face of codimension one. A *pseudomanifold* is a weak pseudomanifold with a connected dual graph. All connected triangulated manifolds are automatically pseudomanifolds.

For any two simplicial complexes X and Y, their *join* $X * Y$ is the simplicial complex whose faces are the disjoint unions of the faces of X with the faces of Y. (Here we adopt the convention that the empty set is a face of every simplicial complex.)

For a finite set α, let $\overline{\alpha}$ (respectively $\partial\alpha$) denote the simplicial complex whose faces are all the subsets (respectively, all proper subsets) of α. Thus, if $\#(\alpha) = n \geq 2$, $\overline{\alpha}$ is a copy of the standard triangulation B_n^{n-1} of the $(n-1)$-dimensional ball, and $\partial\alpha$ is a copy of the standard triangulation S_n^{n-2} of the $(n-2)$-dimensional sphere. Thus, for any two disjoint finite sets α and β, $\overline{\alpha} * \partial\beta$ and $\partial\alpha * \overline{\beta}$ are two triangulations of a ball; they have identical boundaries, namely $(\partial\alpha) * (\partial\beta)$.

A subcomplex Y of a simplicial complex X is said to be an *induced* (or *full*) subcomplex if every face of X contained in the vertex-set of Y is a face of Y. If X is a d-dimensional simplicial complex with an induced subcomplex $\overline{\alpha} * \partial\beta$ ($\alpha \neq \emptyset$, $\beta \neq \emptyset$) of dimension d (thus, $\dim(\alpha) + \dim(\beta) = d$), then $Y := (X \setminus (\overline{\alpha} * \partial\beta)) \cup (\partial\alpha * \overline{\beta})$ is clearly another triangulation of the same topological space $|X|$. In this case, Y is said to be obtained from X by the *bistellar move* $\alpha \mapsto \beta$. If $\dim(\beta) = i$ ($0 \leq i \leq d$), we say that $\alpha \mapsto \beta$ is a *bistellar move of index i* (or an *i-move*, in short). Clearly, if Y is obtained from X by an i-move $\alpha \mapsto \beta$ then X is obtained from Y by the (reverse) $(d-i)$-move $\beta \mapsto \alpha$. Notice that, in case $i = 0$, i.e., when β is a single vertex, we have $\partial\beta = \{\emptyset\}$ and hence $\overline{\alpha} * \partial\beta = \overline{\alpha}$. Therefore, our requirement that $\overline{\alpha} * \partial\beta$ is the induced subcomplex of X on $\alpha \sqcup \beta$ means that β is a new vertex, not in X. Thus, a 0-move creates a new vertex, and correspondingly a d-move deletes an old vertex. For $0 < i < d$, any i-move preserves the vertex-set; these are sometimes called the *proper bistellar moves*. For a thorough treatment of bistellar moves, see [7], for instance.

A triangulation X of a manifold is called a *combinatorial manifold* if its geometric carrier $|X|$ is a piecewise linear (pl) manifold with the pl structure induced from X. A combinatorial triangulation of a sphere/ball is called a *combinatorial sphere/ball* if it induces the standard pl structure (namely, that of the standard sphere/ball) on its geometric carrier. Equivalently (cf. [28, 36]), a simplicial complex is a combinatorial sphere (or ball) if it is obtained from a standard sphere (respectively, a standard ball) by a finite sequence of bistellar moves. In general, a triangulated manifold is a combinatorial manifold if and only if the link of each of its vertices is a combinatorial sphere or combinatorial ball. (Recall that the *link* of a vertex x in a complex X, denoted by $\mathrm{lk}_X(x)$, is the subcomplex $\{\alpha \in X : x \notin \alpha, \alpha \sqcup \{x\} \in X\}$.)

Also, the *star* of x in X, denoted by $\mathrm{st}_X(x)$, is the cone $x * \mathrm{lk}_X(x)$. The *antistar* of x in X, denoted by $\mathrm{ast}_X(x)$, is the subcomplex $\{\alpha \in X : x \notin \alpha\}$.) This leads us to introduce:

Definition 2.1. For $0 \leq k \leq d+1$, a d-dimensional simplicial complex X is said to be *k-stellated* if X may be obtained from S_{d+2}^d by a finite sequence of bistellar moves, each of index $< k$. By convention, S_{d+2}^d is the only 0-stellated simplicial complex of dimension d.

Clearly, for $0 \leq k \leq l \leq d+1$, k-stellated implies l-stellated. All k-stellated simplicial complexes are combinatorial spheres. We let $\Sigma_k(d)$ denote the class of all k-stellated d-spheres. By Pachner's theorem ([36]), $\Sigma_{d+1}(d)$ consists of all combinatorial d-spheres.

By definition, $X \in \Sigma_k(d)$ if and only if there is a sequence $X_0, X_1, \ldots, X_n$ of d-dimensional simplicial complexes such that $X_0 = S_{d+2}^d$, $X_n = X$ and, for $0 \leq j < n$, X_{j+1} is obtained from X_j by a single bistellar move of index $\leq k-1$. The smallest such integer n is said to be the *length* of $X \in \Sigma_k(d)$ and is denoted by $l(X)$. For $X, Y \in \Sigma_k(d)$, we say that Y is shorter than X if $l(Y) < l(X)$. Thus, S_{d+2}^d is the unique shortest member of $\Sigma_k(d)$ (of length 0), and every other member of $\Sigma_k(d)$ can be obtained from a shorter member by a single bistellar move of index $< k$. Thus, induction on the length is a natural method for proving results about the class $\Sigma_k(d)$.

Let X, Y be two pure simplicial complexes of dimension d. We say that X is obtained from Y by the *shelling move* $\alpha \rightsquigarrow \beta$ if α and $\beta \neq \emptyset$ are disjoint faces of X such that (i) $Y \subseteq X$, and $\alpha \sqcup \beta$ is the only facet of X which is not a facet of Y, and (ii) the induced subcomplex of Y on the vertex set of $\alpha \sqcup \beta$ is $\overline{\alpha} * \partial\beta$. If $\dim(\beta) = i$, we say that the shelling move $\alpha \rightsquigarrow \beta$ is of index i. (Clearly, $\dim(\alpha) + \dim(\beta) = d-1$, so that $0 \leq i \leq d$).

We say that a d-dimensional simplicial complex X *shellable* if X is obtained from the standard d-ball B_{d+1}^d by a finite sequence of shelling moves. Clearly, each shelling move increases the number of facets by one, so that - when X is shellable, the number of shelling moves needed to obtain X from B_{d+1}^d is one less than the number of facets of X.

Let X and Y be d-dimensional pseudomanifolds. If X is obtained from Y by the shelling move $\alpha \rightsquigarrow \beta$ then $X = Y \cup \overline{\alpha \sqcup \beta}$, $Y \cap \overline{\alpha \sqcup \beta} = \overline{\alpha} * \partial\beta$. (Since X is a pseudomanifold, it follows that $\overline{\alpha} * \partial\beta \subseteq \partial Y$.) If the move is of index $< d$, then $\overline{\alpha} * \beta$ is a combinatorial $(d-1)$-ball; if it is of index d (so that $\alpha = \emptyset$), $\overline{\alpha} * \partial\beta$ ($= \partial\beta$) is a combinatorial $(d-1)$-sphere. Therefore, if Y is a combinatorial d-ball, then X is also a combinatorial d-ball in case the shelling move is of index $< d$, and X is a combinatorial d-sphere if the shelling move is of index d. (Also note that Y can't be a combinatorial sphere since a d-dimensional pseudomanifold without boundary can't be properly contained in a d-pseudomanifold with or without boundary.) From these observations, it is immediate by an induction on the number of facets that a shellable pseudomanifold of dimension d is either a combinatorial ball or a combinatorial sphere. (This result appears to be due to Danaraj and Klee [13].) Also if X is a shellable d-pseudomanifold, then among the shelling moves used to obtain X from B_{d+1}^d, only the last move can be of index d; this happens if and only if X is a d-sphere. These considerations lead us to introduce:

Definition 2.2. For $0 \leq k \leq d$, a d-dimensional pseudomanifold is said to be *k-shelled* if it may be obtained from the standard d-ball B_{d+1}^d by a finite sequence of shelling moves, each of index $< k$. By convention, B_{d+1}^d is the only 0-shelled pseudomanifold of dimension d.

Clearly, all k-shelled pseudomanifolds are combinatorial balls. Also, for $0 \leq k \leq l \leq d$, k-shelled implies l-shelled. By $\widehat{\Sigma}_k(d)$, $0 \leq k \leq d$, we denote the class of all k-shelled d-balls. Thus $\widehat{\Sigma}_d(d)$ consists of all the shellable d-balls. Note that, while all shellable balls are combinatorial balls, the converse is false.

Unlike the case of bistellar moves, the reverse of a shelling move is not a shelling move. Nonetheless, the two notions are closely related, as the following lemma shows.

Lemma 2.3. *If a triangulated $(d + 1)$-ball X is obtained from a triangulated $(d + 1)$-ball Y by a shelling move $\alpha \rightsquigarrow \beta$ of index $i \leq d$ then the triangulated d-sphere ∂X is obtained from the triangulated d-sphere ∂Y by the bistellar move $\alpha \mapsto \beta$ of index i.*

Proof. Let $\sigma = \alpha \sqcup \beta$. Thus, σ is the only facet of X which is not in Y. Since $Y \subseteq X$ are $(d + 1)$-dimensional pseudomanifolds, it follows that (i) a boundary d-face of Y is not a boundary d-face of X if and only if (it is a face of Y and) it is contained in σ, i.e., if and only if it is a facet of $\overline{\alpha} * \partial\beta$, and (ii) a boundary d face of X is not a face of Y if and only if it is a facet of $\overline{\beta} * \partial\alpha$. Since ∂X and ∂Y are pure simplicial complexes of dimension d, the result follows. $\square$

As an immediate consequence of this lemma, we have:

Corollary 2.4. *If B is a k-shelled $(d + 1)$-ball then ∂B is a k-stellated d-sphere.*

For a simplicial complex X, say of dimension d, and a non-negative integer $m \leq d$, the *m-skeleton* of X, denoted by $\mathrm{skel}_m(X)$, is the subcomplex of X consisting of all its faces of dimension $\leq m$. We recall:

Definition 2.5. For $0 \leq k \leq d + 1$, a triangulated $(d + 1)$-dimensional ball B is said to be *k-stacked* if all the faces of B of codimension (at least) $k + 1$ lie in its boundary; i.e., if $\mathrm{skel}_{d-k}(B) = \mathrm{skel}_{d-k}(\partial B)$. A triangulated d-sphere S is said to be *k-stacked* if there is a k-stacked $(d + 1)$-ball B such that $\partial B = S$. We let $\mathcal{S}_k(d)$ and $\widehat{\mathcal{S}}_k(d)$ denote the class of all k-stacked d-spheres and of all k-stacked d-balls respectively.

Clearly, we have $\mathcal{S}_0(d) \subseteq \mathcal{S}_1(d) \subseteq \cdots \subseteq \mathcal{S}_d(d)$ and $\widehat{\mathcal{S}}_0(d) \subseteq \widehat{\mathcal{S}}_1(d) \subseteq \cdots \subseteq \widehat{\mathcal{S}}_d(d)$. Trivially, the standard d-ball is the only member of $\widehat{\mathcal{S}}_0(d)$, and hence the standard d-sphere is the only member of $\mathcal{S}_0(d)$. Our first proposition shows that $\mathcal{S}_d(d)$ consists of all the triangulated d-spheres. Notice that, trivially, $\widehat{\mathcal{S}}_d(d)$ consists of all triangulated d-balls.

Proposition 2.6. *Every triangulated d-sphere is d-stacked.*

Proof. Let S be a triangulated d-sphere. Fix a vertex x of S. Let A_x be the antistar of x in S. Set $B_x = \overline{\{x\}} * A_x$. It is shown in Lemma 9.1 of [4] that B_x is a triangulated $(d + 1)$-ball. Clearly, B_x has the same vertex set as $S = \partial B_x$. Therefore, B_x is a d-stacked $(d + 1)$-ball and (hence) S is a d-stacked d-sphere. $\square$

Proposition 2.7. *Let B be a triangulated $(d + 1)$-ball. Then B is k-shelled if and only if B is shellable and k-stacked.*

Proof. Suppose B is k-shelled. Then, of course, B is shellable. We prove that B is k-stacked by induction on the number of facets of B. If B has only one facet then $B = B_{d+2}^{d+1}$, the standard ball, and the result is trivial. Otherwise, B is obtained from a k-shelled ball B' (with one less facet) by a single shelling move $\alpha \rightsquigarrow \beta$ of index $\leq k - 1$. By induction hypothesis, $\mathrm{skel}_{d-k}(B') = \mathrm{skel}_{d-k}(\partial B')$, and by Lemma 2.3, ∂B is obtained from $\partial B'$ by the bistellar move $\alpha \mapsto \beta$ of index $\leq k - 1$.

Let γ be a face of B of dimension $\leq d - k$. Since $\dim(\alpha) \geq d - k + 1$, $\gamma \not\supseteq \alpha$. If γ is a face of B' then (as B' is k-stacked), $\gamma \in \partial B'$. Since $\gamma \not\supseteq \alpha$, and ∂B is obtained from $\partial B'$

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by the bistellar move $\alpha \mapsto \beta$, it follows that $\gamma \in \partial B$. If, on the other hand, γ is not a face of B' then $\beta \subseteq \gamma \subseteq \alpha \sqcup \beta$ and hence we have $\gamma \in \overline{\beta} * \partial \alpha \subseteq \partial B$. Thus $\gamma \in \partial B$ in either case. So, B is k-stacked. This proves the "only if" part.

The "if part" is also proved by induction on the number of facets of B. Suppose B is a k-stacked shellable $(d+1)$-ball. If $B = B_{d+2}^{d+1}$, then B is vacuously k-shelled. Else, B is obtained from a shellable $(d+1)$-ball B' (with one less facet) by a single shelling move $\alpha \rightsquigarrow \beta$. By Lemma 2.3, ∂B is obtained from $\partial B'$ by the bistellar move $\alpha \mapsto \beta$. Hence $\alpha \notin \partial B$ but $\alpha \in B$. Since B is k-stacked, it follows that $\dim(\alpha) \geq d - k + 1$, and hence $\dim(\beta) \leq k - 1$. Thus, the shelling move $\alpha \rightsquigarrow \beta$ is of index $\leq k - 1$. Let $\gamma \in B'$, $\dim(\gamma) \leq d - k$. Since $B' \subseteq B$, it follows that $\gamma \in B$. Since $\dim(\gamma) \leq d - k$ and B is k-stacked, it follows that $\gamma \in \partial B$. As $\beta \notin B'$ and $\gamma \in B'$, we also have $\gamma \not\supseteq \beta$. Thus $\gamma \not\supseteq \beta$, $\gamma \in \partial B$ and ∂B is obtained from $\partial B'$ by the bistellar move $\alpha \mapsto \beta$. Hence $\gamma \in \partial B'$. This shows that B' is k-stacked. As B' is k-stacked and shellable, the induction hypothesis implies that B' is k-shelled. Since B is obtained from B' by a shelling move of index $\leq k-1$, it follows that B is also k-shelled. This completes the induction. $\square$

Thus we have $\widehat{\Sigma}_k(d) \subseteq \widehat{\mathcal{S}}_k(d)$. Our next result gives a one-sided relationship between k-stacked spheres and k-stacked balls on one hand, and between k-stellated spheres and k-shelled balls on the other hand.

Proposition 2.8. *Let B be a triangulated ball.*
  (a) *If B is k-stacked then there is a k-stacked sphere S such that B is the antistar of a vertex in S.*
  (b) *If B is k-shelled then there is a k-stellated sphere S such that B is the antistar of a vertex in S.*

Proof. Let x be a new vertex (not in B), and set $S := B \cup (x * \partial B)$. (Notice that, since S is to be a d-pseudomanifold without boundary and B is a d-pseudomanifold with boundary, this is the only choice of S so that B is the antistar of a vertex x in S.) Clearly, $S = \partial B_0$, where $B_0 = x * B$. Therefore, to prove the result, it is enough to show that if B is k-stacked (respectively k-shelled) then so is B_0. But, this is trivial. $\square$

Next we present a characterization of k-stellated spheres of dimension $\geq 2k - 1$.

Proposition 2.9. *A triangulated sphere of dimension $\geq 2k - 1$ is k-stellated if and only if it is the boundary of a k-shelled ball. In consequence, all k-stellated spheres of dimension $\geq 2k - 1$ are k-stacked.*

Proof. The "if" part is Corollary 2.4 (which holds in all dimensions). We prove the "only if" part by induction on the length $l(S)$ of a k-stellated sphere S of dimension $d \geq 2k - 1$. If $l(S) = 0$ then $S = S_{d+2}^d$ is the boundary of B_{d+2}^{d+1}. So, let $l(S) > 0$. Then S is obtained from a shorter member S' of $\Sigma_k(d)$ by a single bistellar move $\alpha \mapsto \beta$ of index $\leq k - 1$. By induction hypothesis, there is a k-shelled $(d+1)$-ball B' such that $\partial B' = S'$. The induced subcomplex of S' on the vertex set $\alpha \sqcup \beta$ is $\overline{\alpha} * \partial \beta \subseteq S' \subseteq B'$. Since $\dim(\beta) \leq k-1 \leq d-k$, $\beta \notin S' = \partial B'$ and (by Proposition 2.7) B' is k-stacked, it follows that $\beta \notin B'$. Thus, the induced subcomplex of B' on $\alpha \sqcup \beta$ is also $\overline{\alpha} * \partial \beta$. So, B' admits the shelling move $\alpha \rightsquigarrow \beta$ of index $\leq k - 1$. Let B be the $(d+1)$-ball obtained from B' by this move. Since B' is k-shelled, so is B. By Lemma 2.3, ∂B is obtained from $S' = \partial B'$ by the bistellar move $\alpha \mapsto \beta$. That is, $\partial B = S$. This completes the induction. The second statement is now immediate from the first statement and Proposition 2.7. $\square$

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Proposition 2.10. *Let S be a k-stacked sphere of dimension $d \geq 2k+1$. Then there is a unique k-stacked ball $\overline{S}$ such that $\partial \overline{S} = S$.*

Proof. Suppose B_1 and B_2 are two k-stacked balls with $\partial B_1 = S = \partial B_2$. Put $S_i = \partial(x*B_i)$, $i = 1, 2$, where x is a new vertex. Then S_1 and S_2 are two $(d+1)$-spheres with identical $(d-k)$-skeletons. Since $d \geq 2k+1$, Theorems 1 and 2 in [14] imply $S_1 = S_2$ and hence $B_1 = B_2$. $\square$

For k-stellated spheres, we can improve the bound in Proposition 2.10. For $d \geq 2k$, such spheres are the boundaries of uniquely determined k-shelled balls. To describe this result, we introduce:

Notation: For a set α and a non-negative integer m, $\binom{\alpha}{\leq m}$ will denote the collection of all subsets of α of size $\leq m$. Also, $\binom{\alpha}{m}$ will denote the collection of all subsets of α of size $= m$

Proposition 2.11. *Let S be a k-stellated sphere of dimension $d \geq 2k$, say with vertex set V. Then there is a unique k-stacked $(d+1)$-ball $\overline{S}$ whose boundary is S. (By Propositions 2.7 and 2.9, $\overline{S}$ is actually k-shelled.) It is given by the formula*

$$\overline{S} = \left\{ \alpha \subseteq V \; : \; \binom{\alpha}{\leq k+1} \subseteq S \right\}. \tag{1}$$

Proof. The existence of a k-stacked ball B with boundary S is guaranteed by Proposition 2.9. We prove that $B = \overline{S}$ by induction on $l(S)$.

If $l(S) = 0$, then $S = S_{d+2}^d$ and trivially B_{d+2}^{d+1} is the unique k-stacked $(d+1)$-ball with boundary S_{d+2}^d; it is indeed given by (1). So, let $l(S) > 0$. Then S is obtained from a shorter member S' of $\Sigma_k(d)$ by a bistellar move $\alpha \mapsto \beta$ of index $\leq k-1$. By induction hypothesis, $\overline{S'}$ (given by (1) with S' in place of S, and the vertex set V' of S' in place of V) is the unique k-stacked ball with boundary S'.

Let B be a k-stacked ball with boundary S. We need to show that $B = \overline{S}$. First we claim that $\alpha \sqcup \beta \in B$. To prove this, fix a vertex $a \in \alpha$ and look at the boundary d-face $\alpha \sqcup \beta \setminus \{a\}$ of B. Let σ be the unique facet of B containing this d-face. Suppose, if possible, that $\sigma \neq \alpha \sqcup \beta$. Let b be the unique vertex in $\sigma \setminus (\alpha \sqcup \beta)$. Then $\beta \cup \{b\} \in B$ and $\dim(\beta \cup \{b\}) \leq k \leq d-k$, so that $\beta \cup \{b\} \in S$. This is a contradiction since $\mathrm{lk}_S(\beta) = \partial\alpha$ and $b \notin \alpha$. This proves the claim: $\alpha \sqcup \beta \in B$.

Put $B' = (B \setminus \overline{\alpha \sqcup \beta}) \cup (\overline{\alpha} * \partial\beta)$. Since $\alpha \sqcup \beta$ is a facet of the triangulated $(d+1)$-ball B, and $\overline{\alpha \sqcup \beta} \cap \partial B = (\partial\alpha) * \overline{\beta}$ is a d-ball, it follows that B' is a triangulated $(d+1)$-ball. Clearly, $\partial B' = S'$. Let γ be a face of B' with $\dim(\gamma) \leq d-k$. Since $B' \subseteq B$, $\gamma \in B$. As B is k-stacked, it follows that $\gamma \in S = \partial B$. Since $\beta \notin B'$, we have $\gamma \not\supseteq \beta$. Therefore, $\gamma \in S' = \partial B'$. Thus B' is a k-stacked $(d+1)$-ball with $\partial B' = S'$. Therefore, by the induction hypothesis, we have $B' = \overline{S'}$. Hence $B = B' \cup \overline{\alpha \sqcup \beta} = \overline{S'} \cup \overline{\alpha \sqcup \beta}$.

So, to complete the proof, we need to show that $B = \overline{S'} \cup \overline{\alpha \sqcup \beta}$ equals $\overline{S}$. Since $d \geq 2k$, it follows that $\mathrm{skel}_k(B) \subseteq \mathrm{skel}_{d-k}(B) \subseteq S$. Therefore, $\sigma \in B \Rightarrow \binom{\sigma}{\leq k+1} \subseteq S \Rightarrow \sigma \in \overline{S}$. Thus, $B \subseteq \overline{S}$. To prove the reverse inclusion $\overline{S} \subseteq B$, let $\sigma \in \overline{S}$. If $\beta \not\subseteq \sigma$ then $\binom{\sigma}{\leq k+1} \subseteq S \Rightarrow \binom{\sigma}{\leq k+1} \subseteq S' \Rightarrow \sigma \in \overline{S'} \subseteq B \Rightarrow \sigma \in B$. So, suppose $\beta \subseteq \sigma$. If $\sigma \not\subseteq \alpha \sqcup \beta$ then take a vertex $x \in \sigma \setminus (\alpha \sqcup \beta)$. Then $\beta \cup \{x\} \in \binom{\sigma}{\leq k+1} \subseteq S \Rightarrow \beta \cup \{x\} \in S$ with $x \notin \alpha$. This is a contradiction since $\mathrm{lk}_S(\beta) = \partial\alpha$. Thus, in this case, $\sigma \subseteq \alpha \sqcup \beta \in B$. Hence $\sigma \in B$ in either case. Thus $\overline{S} \subseteq B$ and hence $B = \overline{S}$. $\square$

Corollary 2.12. *For $k \le e \le d - k - 1$, a k-stellated d-sphere does not have any standard e-sphere as an induced subcomplex. In consequence, such a d-sphere does not admit any bistellar move of index i for $k + 1 \le i \le d - k$.*

Proof. Notice that a triangulated sphere S admits a bistellar move $\alpha \mapsto \beta$ of index i if and only if it has $\overline{\alpha} * \partial \beta$ as an induced subcomplex. In this case, it has the standard $(i - 1)$-sphere $\partial \beta$ as the induced subcomplex on β. So, the second statement is immediate from the first. The first statement is vacuously true unless $d \ge 2k + 1$. So, to prove it, we may assume $d \ge 2k + 1$. If S contains a standard e-sphere as an induced subcomplex on the vertex-set γ (so, $\#(\gamma) = e + 2$), then all the proper subsets of γ are faces of S. In particular, if $e \ge k$, all the subsets of γ of size $\le k + 1$ are faces of S. Hence $\gamma \in \overline{S}$. If, also, $e \le d - k - 1$, then $\gamma \in \mathrm{skel}_{d-k}(\overline{S}) = \mathrm{skel}_{d-k}(S)$ (by Proposition 2.11) and hence $\gamma \in S$. Then the induced subcomplex of S on the vertex set γ is the ball $\overline{\gamma}$, a contradiction. □

If S is a k-stellated d-sphere, other than the standard sphere, then S is obtained from a shorter k-stellated d-sphere by a bistellar move of index $\le k - 1$. Hence such a sphere admits the reverse move, which is a bistellar move of index $\ge d - k + 1$. In consequence, such a sphere always has an induced subcomplex isomorphic to a standard sphere of some dimension $\ge d - k$. In this sense, Corollary 2.12 is best possible. Indeed, it is easy to prove by induction on the length that if $d \ge 2k - 2$ and S is a k-stellated d-sphere which is not $(k - 1)$-stellated, then S has an S^{d-k}_{d-k+2} as an induced subcomplex.

In the following proof (and also later) we use the notation $V(X)$ for the vertex set of a simplicial complex X.

Proposition 2.13. *For a pseudomanifold X, the following are equivalent:*

  (i) X is a 1-shelled ball,

  (ii) X is a 1-stacked ball,

(iii) $\Lambda(X)$ is a tree.

Proof. Let X be of dimension $d + 1 \ge 1$.

  $(i) \Rightarrow (ii)$: Follows from Proposition 2.7.

  $(ii) \Rightarrow (iii)$: The result is trivial for dimension 1. So, assume that $d + 1 \ge 2$. If X has only one facet then the result is trivial. So, assume that X is a 1-stacked ball with at least two facets. Since X is a ball, $\Lambda(X)$ is connected. To prove that $\Lambda(X)$ is a tree, it suffices to show that each edge of $\Lambda(X)$ is a cut edge (i.e., deletion of any edge from $\Lambda(X)$ disconnects the graph). Let $e_0 = \sigma_1 \sigma_2$ be an edge of $\Lambda(X)$. Then $\gamma := \sigma_1 \cap \sigma_2$ is an interior d-face of X; i.e., $\gamma \notin S := \partial X$. Since $\mathrm{skel}_{d-1}(X) = \mathrm{skel}_{d-1}(S)$, $\partial \gamma \subseteq S$. Thus, $\partial \gamma$ is an induced S^{d-1}_{d+1} in the d-sphere S. By Lemma 3.3 of [4], S is obtained from a d-dimensional weak pseudomanifold $\widetilde{S}$ (without boundary) by an elementary handle addition. Since S is simply connected, the Seifert-Van Kampen theorem implies that $\widetilde{S}$ is disconnected and hence has exactly two components (again by Lemma 3.3 of [4]), say S_1 and S_2. Then $S = S_1 \# S_2$ (connected sum) and $V(S_1) \cap V(S_2) = \gamma$. For $1 \le i \le 2$, let U_i be the set of facets of X contained in $V(S_i)$. Since $V(S_1) \cap V(S_2) = \gamma$, it follows that $U_1 \cap U_2 = \emptyset$.

If the dimension $d + 1 = 2$ then γ is an edge and it clearly divides the 2-disc X into two parts and the triangles (facets) in one part are in U_1 and the triangles in the other part are in U_2. Now, assume that $d + 1 \ge 3$. Let uv be an edge of X. Since $d + 1 \ge 3$, $uv \in S$ and hence (since $S = S_1 \# S_2$) $uv \in S_1$ or $uv \in S_2$. Therefore, $u, v \in V(S_1)$ or $u, v \in V(S_2)$.

This implies that for any facet σ in X, either all the vertices of σ are in $V(S_1)$ or all the vertices of σ are in $V(S_2)$. Thus, any facet in X is in U_1 or in U_2. Thus (for any dimension $d + 1 \geq 2$), $U_1 \sqcup U_2$ is a partition of the vertex-set of the dual graph $\Lambda(X)$. Any facet σ of X containing a d-face $\alpha \neq \gamma$ of S_1 is in U_1. So, $U_1 \neq \emptyset$. Similarly, $U_2 \neq \emptyset$.

Now, let $e = \alpha_1\alpha_2$ be an edge of $\Lambda(X)$ with $\alpha_i \in U_i$, $i = 1, 2$. Then $\alpha := \alpha_1 \cap \alpha_2 \subseteq V(S_i)$ for $i = 1, 2$. Hence $\alpha \subseteq V(S_1) \cap V(S_2) = \gamma$ and therefore $\alpha = \gamma$. So, $e = e_0$. Thus, e_0 is the unique edge of $\Lambda(X)$ with one end in U_1 and other end in U_2. So, e_0 is a cut edge of $\Lambda(X)$. Since e_0 was an arbitrary edge of $\Lambda(X)$, this proves that $\Lambda(X)$ is a tree.

$(iii) \Rightarrow (i)$: Suppose $\Lambda(X)$ is a tree. We prove that X is 1-shelled by induction on the number of facets of X (i.e., the number of vertices of $\Lambda(X)$). This is trivial if X has only one facet, i.e., $X = B_{d+2}^{d+1}$. So, assume $\Lambda(X)$ is a tree with at least two vertices. Then $\Lambda(X)$ has a vertex σ of degree 1 (an end vertex). Let σ' be the unique neighbour of σ in $\Lambda(X)$, and put $\gamma = \sigma \cap \sigma'$. Let $X' = (X \setminus \overline{\sigma}) \cup \overline{\gamma}$. Then X' is a pseudomanifold and $\Lambda(X')$ is the tree obtained from the tree $\Lambda(X)$ by deleting the end vertex σ and the edge $\sigma\sigma'$. Therefore, by induction hypothesis, X' is an 1-shelled ball. If u is the vertex of X in $\sigma \setminus \gamma$, then X is obtained from X' by the shelling move $\gamma \rightsquigarrow \{u\}$ of index 0. Therefore, X is also an 1-shelled ball. $\qquad \square$

Thus a triangulated ball is 1-stacked if and only if it is 1-shelled. So, $\widehat{\Sigma}_1(d) = \widehat{\mathcal{S}}_1(d)$. Now, Propositions 2.9 and 2.13 imply:

Corollary 2.14. *A triangulated sphere is 1-stellated if and only if it is 1-stacked.*

Next we introduce:

Definition 2.15. For $0 \leq k \leq d$, $\mathcal{W}_k(d)$ consists of the connected simplicial complexes of dimension d all whose vertex-links are k-stellated $(d-1)$-spheres, and $\mathcal{K}_k(d)$ consists of the connected simplicial complexes of dimension d all whose vertex-links are k-stacked $(d-1)$-spheres.

Thus, members of $\mathcal{W}_k(d)$ are combinatorial manifolds; the members of $\mathcal{K}_k(d)$ are triangulated manifolds. In consequence of Corollary 2.14, we have:

Corollary 2.16. $\mathcal{W}_1(d) = \mathcal{K}_1(d)$.

In consequence of Proposition 2.9, we have:

Corollary 2.17. $\mathcal{W}_k(d) \subseteq \mathcal{K}_k(d)$ *for* $d \geq 2k$.

Proposition 2.18. *(a) All k-stellated d-spheres belong to the class $\mathcal{W}_k(d)$. (b) All k-stacked d-spheres belong to the class $\mathcal{K}_k(d)$.*

Proof. Let S be a k-stellated d-sphere. We need to show that all the vertex-links of S are k-stellated. Again, the proof is by induction on the length $l(S)$ of S. If $l(S) = 0$ then $S = S_{d+2}^d$, and all its vertex links are S_{d+1}^{d-1}, so we are done. Therefore, let $l(S) > 0$. Then S is obtained from a shorter k-stellated d-sphere S' by a bistellar move $\alpha \mapsto \beta$ of index $\leq k - 1$. Let x be a vertex of S. If $x \notin \alpha \sqcup \beta$ then $\text{lk}_S(x) = \text{lk}_{S'}(x)$ is k-stellated by induction hypothesis. If $x \in \alpha$ then $\text{lk}_S(x)$ is obtained from the k-stellated sphere $\text{lk}_{S'}(x)$ by the bistellar move $\alpha \setminus \{x\} \mapsto \beta$ of index $\leq k - 1$. If $x \in \beta$ and $\beta \neq \{x\}$ then $\text{lk}_S(x)$ is obtained from the k-stellated sphere $\text{lk}_{S'}(x)$ by the bistellar move $\alpha \mapsto \beta \setminus \{x\}$ of index

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$\leq k - 2$. If $\beta = \{x\}$ then $\mathrm{lk}_S(x)$ is the standard sphere $\partial\alpha$. Thus, in all cases, $\mathrm{lk}_S(x)$ is k-stellated. This proves part (a).

Let S be a k-stacked d-sphere. Let B be a k-stacked $(d+1)$-ball such that $\partial B = S$. If x is a vertex of S then x is a vertex of B and $B' = \mathrm{lk}_B(x)$ is a d-ball with $\partial B' = \mathrm{lk}_S(x)$. Therefore, it suffices to show that B' is also k-stacked. Indeed, if γ is a face of B' of codimension $\geq k+1$ then $\gamma \cup \{x\}$ is a face of B of codimension $\geq k+1$, and hence $\gamma \cup \{x\} \in \partial B = S$, so that $\gamma \in \mathrm{lk}_S(x) = \partial B'$. $\qquad\square$

Proposition 2.19. *Let $d \geq 2k + 2$ and $M \in \mathcal{W}_k(d)$. Let $V(M)$ be the vertex set of M. Then*

$$\overline{M} := \left\{ \alpha \subseteq V(M) \;:\; \binom{\alpha}{\leq k+2} \subseteq M \right\} \tag{2}$$

is the unique combinatorial $(d+1)$-manifold such that $\partial\overline{M} = M$ and $\mathrm{skel}_{d-k}(\overline{M}) = \mathrm{skel}_{d-k}(M)$.

Proof. Fix $x \in V(\overline{M}) = V(M)$.

Claim: $\mathrm{lk}_{\overline{M}}(x) = \overline{\mathrm{lk}_M(x)}$, where the right hand side is as defined in Proposition 2.11.

From the definition, we see that $\alpha \in \mathrm{lk}_{\overline{M}}(x) \Rightarrow \alpha \sqcup \{x\} \in \overline{M} \Rightarrow \binom{\alpha \sqcup \{x\}}{\leq k+2} \subseteq M \Rightarrow \binom{\alpha}{\leq k+1} \subseteq \mathrm{lk}_M(x) \Rightarrow \alpha \in \overline{\mathrm{lk}_M(x)}$. Thus, we have $\mathrm{lk}_{\overline{M}}(x) \subseteq \overline{\mathrm{lk}_M(x)}$.

Conversely, let $\alpha \in \overline{\mathrm{lk}_M(x)}$. Then $\binom{\alpha}{\leq k+1} \subseteq \mathrm{lk}_M(x)$, so that each $\gamma \subseteq \alpha \sqcup \{x\}$ such that $x \in \gamma$ and $\#(\gamma) \leq k+2$ is in M. Therefore, to prove that $\alpha \in \mathrm{lk}_{\overline{M}}(x)$, it suffices to show that each $\gamma \subseteq \alpha$ with $\#(\gamma) \leq k+2$ is in M. Since $\alpha \in \overline{\mathrm{lk}_M(x)}$, such a set γ is in $\overline{\mathrm{lk}_M(x)}$, and hence $\gamma \in \mathrm{skel}_{k+1}(\overline{\mathrm{lk}_M(x)}) \subseteq \mathrm{skel}_{d-k-1}(\overline{\mathrm{lk}_M(x)}) = \mathrm{skel}_{d-k-1}(\mathrm{lk}_M(x)) \subseteq \mathrm{lk}_M(x) \subseteq M$. (Here the first inclusion holds since $k+1 \leq d-k-1$.) This proves that $\alpha \in \overline{\mathrm{lk}_M(x)} \Rightarrow \alpha \in \mathrm{lk}_{\overline{M}}(x)$, so that $\overline{\mathrm{lk}_M(x)} \subseteq \mathrm{lk}_{\overline{M}}(x)$. This proves the claim.

In view of Proposition 2.11, the claim implies that $\overline{M}$ is a combinatorial $(d+1)$-manifold with boundary, and $\mathrm{lk}_{\partial\overline{M}}(x) = \partial(\mathrm{lk}_{\overline{M}}(x)) = \partial(\overline{\mathrm{lk}_M(x)}) = \mathrm{lk}_M(x)$ for every vertex x. Therefore, $\partial\overline{M} = M$, and we have:

$$
\begin{aligned}
\mathrm{lk}_{\mathrm{skel}_{d-k}(\overline{M})}(x) &= \mathrm{skel}_{d-k-1}(\mathrm{lk}_{\overline{M}}(x)) = \mathrm{skel}_{d-k-1}(\overline{\mathrm{lk}_M(x)}) = \mathrm{skel}_{d-k-1}(\mathrm{lk}_M(x)) \\
&= \mathrm{lk}_{\mathrm{skel}_{d-k}(M)}(x)
\end{aligned}
$$

for every vertex x. Thus, $\mathrm{skel}_{d-k}(\overline{M}) = \mathrm{skel}_{d-k}(M)$.

Now, if N is any $(d+1)$-manifold with $\partial N = M$ and $\mathrm{skel}_{d-k}(N) = \mathrm{skel}_{d-k}(M)$, then for any vertex x, we have:

$$
\begin{aligned}
\partial(\mathrm{lk}_N(x)) &= \mathrm{lk}_{\partial N}(x) = \mathrm{lk}_M(x), \quad \text{and} \\
\mathrm{skel}_{d-k-1}(\mathrm{lk}_N(x)) &= \mathrm{lk}_{\mathrm{skel}_{d-k}(N)}(x) = \mathrm{lk}_{\mathrm{skel}_{d-k}(M)}(x) = \mathrm{skel}_{d-k-1}(\mathrm{lk}_M(x)).
\end{aligned}
$$

Therefore, the uniqueness assertion in Proposition 2.11 implies that $\mathrm{lk}_N(x) = \overline{\mathrm{lk}_M(x)} = \mathrm{lk}_{\overline{M}}(x)$ for every vertex x and hence $N = \overline{M}$. This completes the proof. $\qquad\square$

Remark 2.20. If M is a k-stellated sphere of dimension $d \geq 2k+2$ then $M \in \mathcal{W}_k(d)$ by Proposition 2.18. In this case, the uniqueness statements in Propositions 2.11 and 2.19 show that the two definitions of $\overline{M}$ (given in (1) and (2)) agree. Also, if we define $\overline{\mathcal{W}}_k(d+1)$ to be the class of all $(d+1)$-dimensional simplicial complexes all whose vertex links are k-shelled d-balls, then by Propositions 2.11 and 2.19, for $d \geq 2k+2$, $M \mapsto \overline{M}$ is a bijection from $\mathcal{W}_k(d)$ onto $\overline{\mathcal{W}}_k(d+1)$. The boundary map provides its inverse.

3   Polytopal spheres and balls : a diversion

For a subset A of an Euclidean space, we write $\mathrm{conv}(A)$ (respectively $\mathrm{aff}(A)$) for the convex (respectively affine) hull of A. For a convex set C, the topological interior (respectively the topological boundary) of C in $\mathrm{aff}(C)$ is called the *relative interior* (respectively the *boundary*) of C and we denote it by C° (respectively $C^{\bullet}$).

Recall that a (convex) *polytope* in the Euclidean space $\mathbb{R}^n$ is the convex hull of a finite set of points. Equivalently, a polytope in $\mathbb{R}^n$ is a compact subset of $\mathbb{R}^n$ which may be obtained as the intersection of finitely many closed half-spaces of $\mathbb{R}^n$. As general references on polytopes, cf [20, 42]. The *dimension* of a polytope P is defined to be the dimension of the affine space $\mathrm{aff}(P)$. A (geometric) *simplex* is a polytope which is the convex hull of a set of affinely independent points. A *face* of a polytope P in $\mathbb{R}^n$ is either P itself or is the intersection of P with a hyperplane H of $\mathbb{R}^n$ such that P is contained in one of the two closed half spaces determined by H. The zero-dimensional faces of a polytope are called its *vertices*, and the d-dimensional faces (i.e., maximal proper faces) of a $(d+1)$-dimensional polytope are called its *facets*. Notice that any polytope is the convex hull of its vertex set. It is also the disjoint union of the relative interiors of its faces.

Recall that a *geometric simplicial complex* X is a collection of geometric simplices such that the intersection of any two members of X is again a member of X and any face of a member of X is again a member of X. If X is a geometric simplicial complex with vertex set $V(X)$, then $X_{\mathrm{abs}} := \{A \subseteq V(X) : \mathrm{conv}(A) \in X\}$ is an abstract simplicial complex and is called the *abstract scheme* of X. We sometimes identify X with X_{abs}.

A polytope is *simplicial* if all its proper faces are simplices. If P is a simplicial polytope then all the proper faces of P form a geometric simplicial complex $\mathrm{Bd}(P)$. The abstract scheme of $\mathrm{Bd}(P)$ is called the *boundary complex* of P and is denoted by ∂P. Thus, $\partial P = \{A \subseteq V(P) : \mathrm{conv}(A) \text{ is a proper face of } P\}$. Clearly, the union of all the proper faces of P is the topological boundary $P^{\bullet}$ of P. Thus, the boundary complex ∂P of P triangulates the topological sphere $P^{\bullet}$. We identify $\mathrm{Bd}(P)$ with ∂P.

Definition 3.1. A triangulated sphere is said to be a *polytopal sphere* if it is isomorphic to the boundary complex of a simplicial polytope.

Definition 3.2. A *simplicial subdivision* P' of a simplicial polytope P is a geometric simplicial complex such that $V(P') = V(P)$ and P is the union of all the simplices in P'. Let $\overline{P}$ be the abstract scheme of P'. Then $\overline{P}$ triangulates P and hence $\overline{P}$ is a triangulated ball. We identify $\overline{P}$ with P' and also say that $\overline{P}$ is a *simplicial subdivision* of P. A *polytopal d-ball* is a triangulated d-ball which is isomorphic to a simplicial subdivision $\overline{P}$ of some d-polytope P.

(Warning : Most authors do not include the hypothesis $V(P') = V(P)$ in the definition of simplicial subdivision.)

Lemma 3.3. *Let P be a polytope with vertex set $V(P)$. Let $A \subseteq V(P)$. Then*

  (a) *either $\mathrm{conv}(A) \subseteq P^{\bullet}$ or $\mathrm{conv}(A)^{\circ} \subseteq P^{\circ}$; and*

  (b) *if, further, P is simplicial and $\mathrm{conv}(A) \subseteq P^{\bullet}$ then $\mathrm{conv}(A)$ is a proper face of P.*

Proof. (a) Suppose $\mathrm{conv}(A) \not\subseteq P^{\bullet}$. Then there is a point u_0 in $\mathrm{conv}(A) \cap P^{\circ}$. For any point $x \in \mathrm{conv}(A)^{\circ}$, $x \neq u_0$, the line L joining x and u_0 meets $\mathrm{conv}(A)$ in a line segment

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$[a, b]$. Since x belongs to $\text{conv}(A)^{\circ}$, it belongs to the relative interior (a, b) of the segment $[a, b]$. Also, if $L \cap P = [c, d]$ then $[a, b] \subseteq [c, d]$. Since $u_0 \in L \cap P^{\circ}$, it follows that $(c, d) \subseteq P^{\circ}$. Thus, $x \in (a, b) \subseteq (c, d) \subseteq P^{\circ}$. So, $\text{conv}(A)^{\circ} \subseteq P^{\circ}$.

(b) We may assume that $\text{aff}(P) = \mathbb{R}^d$. In this case $\text{conv}(A)$ and P° are disjoint convex sets, of which the first one is compact and the second one is open in $\mathbb{R}^d$. So, there is a hyperplane H in $\mathbb{R}^d$ strictly separating $\text{conv}(A)$ and P°. Then $\text{conv}(A)$ is contained in the proper face $H \cap P$ of P. Since P is simplicial, it follows that $\text{conv}(A)$ is a face of P. □

Notice that, as a consequence of Lemma 3.3, if P' is a simplicial subdivision (in the sense of Definition 3.2, which is stronger than the usual definition) of a simplicial d-polytope P, then each simplex in $\partial P'$ is a proper face of P and hence (since both $\partial P'$ and ∂P are $(d-1)$-pseudomanifolds without boundary) $\partial P' = \partial P$. Thus, if B is a polytopal ball triangulating a simplicial polytope P, then ∂B is isomorphic to the boundary complex ∂P of P.

The following proposition is essentially Theorem 4.1 in [33].

Proposition 3.4. *Let B be a k-stacked triangulated ball of dimension $d + 1 \geq 2k + 1$. If ∂B is a polytopal d-sphere then B is a polytopal ball.*

Proof. Since ∂B is polytopal, there is a $(d + 1)$-polytope P in $\mathbb{R}^{d+1}$ such that ∂B is the boundary complex ∂P of P. Thus, we may identify the vertices of B with those of P. For any face $\alpha \in B$, let $|\alpha|$ denote the convex hull of α. Note that, for $\alpha \in \partial B$, $|\alpha|$ is a proper face of P. It follows that the simplices $|\alpha|$, $\alpha \in \partial B$, have pairwise disjoint relative interiors. Indeed, Lemma 3.3 implies that, for $\alpha \in B$ and $\beta \in \partial B$, $|\alpha|$ and $|\beta|$ have disjoint relative interiors, whenever $\alpha \neq \beta$.

Claim 1: If α is an i-face of B then $|\alpha|$ is a geometric (non-singular) i-simplex. That is, $\dim(|\alpha|) = \dim(\alpha)$ for all $\alpha \in B$.

Suppose there exists an i-face α of B such that $|\alpha|$ is not a geometric i-simplex. Then α is a set of $i+1$ points in the affine space $\text{aff}(\alpha)$ of dimension $\leq i-1$. Then, by Radon's Theorem (cf. [20, Page 124]), there exist disjoint proper subsets $\beta, \gamma \subseteq \alpha$ such that $|\beta|^{\circ} \cap |\gamma|^{\circ} \neq \emptyset$. Let $\#(\gamma) \leq \#(\beta)$. Then, $2\#(\gamma) \leq \#(\alpha) \leq d + 2 \leq 2d - 2k + 2$. Thus, $\#(\gamma) \leq d - k + 1$. So, $\dim(\gamma) \leq d - k$. Therefore, $\gamma \in \partial B$ and hence (by the comment preceding Claim 1) $|\beta|^{\circ} \cap |\gamma|^{\circ} = \emptyset$, a contradiction. This proves Claim 1.

Claim 2: $X := \{|\alpha| : \alpha \in B\}$ is a geometric simplicial complex.

We have to show that for any two faces α, β in B, $|\alpha| \cap |\beta|$ is a common face of both $|\alpha|$ and $|\beta|$. Otherwise, $I := \{(\alpha, \beta) \in B \times B : |\alpha| \cap |\beta| \text{ is not a common face of } |\alpha| \text{ and } |\beta|\}$ is a non-empty set. Fix $(\alpha, \beta) \in I$ such that $\dim(\alpha) + \dim(\beta)$ is minimum. If $|\alpha| \cap |\beta|^{\circ} = \emptyset$ then there is a proper face β_1 of β such that $|\alpha| \cap |\beta_1| = |\alpha| \cap |\beta|$ and hence $(\alpha, \beta_1) \in I$, contradicting the choice of (α, β). So, $|\alpha| \cap |\beta|^{\circ} \neq \emptyset$. Similarly, $|\alpha|^{\circ} \cap |\beta| \neq \emptyset$. Therefore $|\alpha|^{\circ} \cap |\beta|^{\circ} \neq \emptyset$ (if $x \in |\alpha| \cap |\beta|^{\circ}$, $y \in |\alpha|^{\circ} \cap |\beta|$ then $\frac{x+y}{2} \in |\alpha|^{\circ} \cap |\beta|^{\circ}$). Hence $\alpha, \beta \in B \setminus \partial B$. Since B is k-stacked, it follows that $\dim(\alpha) \geq d - k + 1$, $\dim(\beta) \geq d - k + 1$. Hence $\dim(\alpha) + \dim(\beta) \geq 2d - 2k + 2 \geq d + 2$. Since $|\alpha|, |\beta|$ are simplices in $\mathbb{R}^{d+1}$, it follows that there is a line L, through any given point $x \in |\alpha|^{\circ} \cap |\beta|^{\circ}$, contained in $\text{aff}(\alpha) \cap \text{aff}(\beta)$. Since the line segments $L \cap |\alpha|$ and $L \cap |\beta|$ have the interior point x in common, it follows that $[a, b] := L \cap |\alpha| \cap |\beta|$ is a non-trivial line segment. Thus, $a \neq b$ are points in the boundary of $|\alpha| \cap |\beta|$. If $a \in |\alpha|^{\circ}$ then $a \in |\beta|^{\bullet}$ and hence $a \in |\beta_1|^{\circ}$ for some proper face β_1 of β. Then $(\alpha, \beta_1) \in I$, contradicting the choice of (α, β). So, $a \in |\alpha|^{\bullet}$. Similarly, $b \in |\alpha|^{\bullet}$. Since $x \in [a, b] \cap |\alpha|^{\circ}$, a, b are not both vertices of α. Assume that a is not a vertex of α. Since

$a \in |\alpha|^{\bullet}$, $a \in |\alpha_1|^{\circ}$ for some proper face α_1 (of dimension ≥ 1) of α. Then $(\alpha_1, \beta) \in I$, contradicting the choice of (α, β). This completes the proof of Claim 2.

Claim 1 shows that B is the abstract scheme of X. Let $|B|$ denote the union of the simplices in X. Since B is a triangulated ball, it follows that $|B|$ is a topological $(d+1)$-ball. Clearly $|B| \subseteq P$. Since $|B|$ and P are topological $(d+1)$-balls with the same boundary, $|B| = P$. So, X is a simplicial subdivision of P and is abstractly isomorphic to B. $\qquad\square$

Proposition 3.5. *Let B be a polytopal k-stacked ball. Then B does not contain any standard sphere of dimension $\geq k$ as an induced subcomplex.*

Proof. Let $\dim(B) = d$. We may assume that B is (the abstract scheme of) a simplicial subdivision of a simplicial d-polytope P. For any set A of vertices of B, we let $\langle A \rangle$ denote the convex hull of A.

Suppose, if possible, that α is the vertex set of an induced S^m_{m+2} in B, $m \geq k$. Since the d-pseudomanifold B can't properly contain a d-pseudomanifold without boundary, we must have $k \leq m < d$. Let β be an m-face of B contained in α. Clearly, there is a facet σ of B containing β such that $\langle\sigma\rangle^{\circ} \cap \langle\alpha\rangle^{\circ} \neq \emptyset$. Write $\sigma = \beta \sqcup \mu$. Take a point x in $\langle\sigma\rangle^{\circ} \cap \langle\alpha\rangle^{\circ}$. Since $x \in \langle\beta \sqcup \mu\rangle^{\circ}$, there are points $b \in \langle\beta\rangle^{\circ}$, $c \in \langle\mu\rangle^{\circ}$ such that x belongs to the open line segment (b, c). Let the line bc meet $\langle\alpha\rangle$ in the line segment $[b, b']$. Since b' is in the boundary of $\langle\alpha\rangle$, there is a face $\gamma \subset \alpha$, $\gamma \nsubseteq \beta$, such that $b' \in \langle\gamma\rangle^{\circ}$. Since we have the point x in $(b, c) \cap (b, b')$, it follows that either $b' \in (b, c)$ or $b' = c$ or $c \in (b, b')$. If $b' \in (b, c)$ then $b' \in \langle\sigma\rangle^{\circ}$ and hence $b' \in \langle\sigma\rangle^{\circ} \cap \langle\gamma\rangle^{\circ}$. So, $\langle\sigma\rangle^{\circ} \cap \langle\gamma\rangle^{\circ} \neq \emptyset$. But, this is not possible since $\sigma \neq \gamma$ are faces of B. If $b' = c$ then $b' \in \langle\mu\rangle^{\circ}$ and hence $\langle\mu\rangle^{\circ} \cap \langle\gamma\rangle^{\circ} \neq \emptyset$. This is also not possible since $\mu \neq \gamma$ are faces of B. Therefore, $c \in (b, b')$. By Lemma 3.3, $\langle\alpha\rangle^{\circ} \subseteq P^{\circ}$. Now, c belongs to $\langle\alpha\rangle^{\circ} \subseteq P^{\circ}$ as well as to $\langle\mu\rangle^{\circ}$. Thus, $\langle\mu\rangle^{\circ} \cap P^{\circ} \neq \emptyset$. Then, by Lemma 3.3, $\langle\mu\rangle^{\circ} \subseteq P^{\circ}$ and hence $\mu \notin \partial P = \partial B$. This is a contradiction since $\mu \in B$, B is a k-stacked d-ball and $\dim(\mu) = d - m - 1 \leq d - k - 1$. $\qquad\square$

Corollary 3.6. *Let S be a k-stacked polytopal sphere of dimension $d \geq 2k$. Then there is a unique k-stacked $(d+1)$-ball $\overline{S}$ such that $S = \partial\overline{S}$. Further, $\overline{S}$ is given by the formula (1).*

Proof. Let B be a k-stacked $(d+1)$-ball such that $\partial B = S$. Then, by Proposition 3.4, B is a polytopal ball. Therefore, by Proposition 3.5, B contains no induced standard sphere of dimension $\geq k$. Since B is a k-stacked ball of dimension $\geq 2k + 1$, $\mathrm{skel}_k(B) \subseteq \partial B = S$. Therefore, $B \subseteq \overline{S}$, where $\overline{S}$ is defined by formula (1). If $B \neq \overline{S}$ then take a minimal face $\alpha \in \overline{S} \setminus B$. Since $S \subseteq B$, we have $\alpha \in \overline{S} \setminus S$, and hence $\dim(\alpha) \geq k + 1$. Therefore, α induces a standard sphere of dimension $\geq k$ in B, a contradiction. Thus $B = \overline{S}$. $\qquad\square$

Remark 3.7. The uniqueness statement in Corollary 3.6 is due to McMullen ([33, Theorem 3.3]). However, the explicit description of the ball $\overline{S}$ given above appears to be new. Indeed, after his Theorem 4.1, McMullen remarks: "The implication of Theorem 4.1 for the equality case (Conjecture 3.1) of GLBC is obvious - all we need is an appropriate combinatorial triangulation of our polytope P. However, there are no corresponding pointers to finding such a triangulation." Corollary 3.6 provides such "pointers", answering the question implicit in McMullen's remark.

Proposition 3.8. *Let S be a $(k+1)$-neighbourly polytopal sphere of dimension d. Then S is $(d-k)$-stellated.*

Proof. Fix a vertex x of S. Let A be the antistar of x in S. By Bruggesser-Mani (cf. [42, Theorem 8.12]) A is a shellable d-ball. Hence, $x * A$ is a shellable $(d + 1)$-ball. Clearly, $\partial(x * A) = S$. Since S is $(k+1)$-neighbourly, $x * A$ is $(d-k)$-stacked. Hence, by Proposition 2.7, $x * A$ is $(d - k)$-shelled. Therefore, by Corollary 2.4, S is $(d - k)$-stellated. $\square$

Notice that, as a particular case of Proposition 3.8, any $(k + 1)$-neighbourly polytopal sphere of dimension $2k + 1$ is $(k+1)$-stellated. Also, every polytopal d-sphere is d-stellated.
The case $k = 1$ of the following result is due to M. A. Perles (cf. [1, Theorem 1]).

Proposition 3.9. *Let S be a $(k + 1)$-neighbourly polytopal sphere of dimension $2k + 1$. Then $S \in \mathcal{W}_k(2k + 1)$.*

Proof. Let S be the boundary complex of a simplicial polytope P. Then P is a $(k + 1)$-neighbourly $(2k + 2)$-polytope. Fix a vertex v of S, and let $L = \mathrm{lk}_S(v)$. We need to prove that L is k-stellated. This is trivial if S is a standard sphere. So, assume that P is not a simplex. It follows that $Q := \mathrm{conv}(V(P) \setminus \{v\})$ is also a $(2k + 2)$-dimensional polytope. Clearly, Q is also $(k+1)$-neighbourly and hence, by Radon's Theorem, Q is also simplicial. Let B be the pure simplicial complex of dimension $2k + 1$ whose facets are those facets of the polytope Q which are visible from the point v. By Bruggesser-Mani (cf. [42, Theorem 8.12]), B is a shellable ball. Clearly, $\partial B = L = S \cap B$. Let α be a k-face of B. Then (as S is $(k + 1)$-neighbourly) $\alpha \in S \cap B = \partial B$. Thus, B is k-stacked. Hence $L = \mathrm{lk}_S(v)$ is k-stellated by Propositions 2.7 and 2.9. Since v was an arbitrary vertex of S, it follows that $S \in \mathcal{W}_k(2k + 1)$. $\square$

4 The *g*-, beta- and mu-vectors and tightness

For a d-dimensional simplicial complex X, $f_i = f_i(X)$ denotes the number of i-dimensional faces of X $(-1 \leq i \leq d)$. Thus, $f_{-1} = 1$, corresponding to the empty face of X. The vector $\boldsymbol{f}(X) = (f_0, f_1, \ldots, f_d)$ is called the *face-vector* (or *f-vector*) of X.

We recall that a pure d-dimensional simplicial complex X is shellable if it may be obtained from the standard d-ball by a finite sequence of shelling moves. Thus, X is shellable if there is a shelling sequence $B_{d+1}^d = X_0 \subset X_1 \subset \cdots \subset X_n = X$ of (necessarily pure) simplicial complexes such that, for $0 \leq i \leq n-1$, X_{i+1} is obtained from X_i by a single shelling move. Clearly, each shelling move of index $j - 1$ increases the number of i-faces of a d-dimensional simplicial complex by $\binom{d-j+1}{i-j+1}$. Therefore, if, amongst a sequence of shelling moves used to obtain X from B_{d+1}^d, exactly h_j are of index $j - 1$ $(0 \leq j \leq d+1)$, then the face-vector of X is given by

$$f_i(X) = \sum_{j=0}^{i+1} \binom{d - j + 1}{i - j + 1} h_j, \quad -1 \leq i \leq d.$$

(Here, by convention, $h_0 = 1$, and the term with $j = 0$ gives the number of i-faces in the initial standard d-ball.)

Inverting this system of linear equations, we find that the numbers h_j are given in terms of the face-vector of X by the formula

$$h_j = \sum_{i=-1}^{j-1} (-1)^{j-i-1} \binom{d - i}{j - i - 1} f_i, \quad 0 \leq j \leq d+1. \tag{3}$$

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This formula shows that the vector $\boldsymbol{h}(X) = (h_0, \ldots, h_{d+1}) = (h_0(X), \ldots, h_{d+1}(X))$ depends only on the simplicial complex X, and not on the particular sequence of shelling moves used to obtain X. It is called the h-vector of X. More generally, for any simplicial complex X of dimension d, the h-vector of X is defined in terms of its f-vector by the formula (3).

The g-vector $\boldsymbol{g}(X) = (g_0, g_1, \ldots, g_{d+1}) = (g_0(X), g_1(X), \ldots, g_{d+1}(X))$ of a simplicial complex X of dimension d is defined in terms of its h-vector by the formula

$$g_j(X) = h_j(X) - h_{j-1}(X), \quad 0 \le j \le d+1$$

(where $h_{-1}(X) \equiv 0$). In view of (3), the g-vector of X is given in terms of its f-vector by:

$$g_j(X) = \sum_{i=-1}^{j-1} (-1)^{j-i-1} \binom{d-i+1}{j-i-1} f_i(X), \quad 0 \le j \le d+1. \tag{4}$$

(4) may be inverted to obtain the f-vector of X in terms of its g-vector:

$$f_i(X) = \sum_{j=0}^{i+1} \binom{d-j+2}{i-j+1} g_j(X), \quad -1 \le i \le d. \tag{5}$$

Let $B_{d+1}^d = X_0 \subset X_1 \subset \cdots \subset X_n = S$ be a shelling sequence for a shellable d-sphere S. For $0 \le j \le n$, let $\widetilde{X}_j$ be the pure simplicial complex of dimension d whose facets are those facets of S which are not in X_{j-1} (with $X_{-1} = \emptyset$). Then $B_{d+1}^d = \widetilde{X}_n \subset \widetilde{X}_{n-1} \subset \cdots \subset \widetilde{X}_0 = S$ is another shelling sequence for S. Indeed, if X_j is obtained from X_{j-1} by the shelling move $\alpha_j \rightsquigarrow \beta_j$ then $\widetilde{X}_{j-1}$ is obtained from $\widetilde{X}_j$ by the shelling move $\beta_j \rightsquigarrow \alpha_j$. Therefore, if the original shelling sequence for S involves h_i shelling moves of index $i-1$, then the reverse sequence involves h_i shelling moves of index $d-i$. Since the h-vector of S is independent of the particular shelling sequence used, this shows that the h-vector of any shellable d-sphere satisfies $h_{d+1-i} = h_i$, $0 \le i \le d+1$. From the definition $g_i := h_i - h_{i-1}$, it follows that the g-vector of any shellable d-sphere satisfies

$$g_{d+2-i} = -g_i, \quad 1 \le i \le d+1. \tag{6}$$

In fact, the g-vector of any triangulated d-sphere satisfies (6). Indeed, this is equivalent to the famous Dehn-Sommerville equations for triangulated spheres. Even more generally, the g-vector of any triangulated closed d-manifold with Euler characteristic χ satisfies Klee's formula (cf. [23]):

$$g_{d+2-i} + g_i = (-1)^{i-1} \binom{d+2}{i} (\chi - \chi(S^d)), \quad 1 \le i \le d+1. \tag{7}$$

(In particular, any triangulated closed manifold of odd dimension d satisfies (6).) However, for k-stellated d-spheres, the g-vector has a geometric significance which is lacking in the more general situations. This geometric meaning of the g-vector stems from:

Lemma 4.1. *Let X, Y be two simplicial complexes of dimension d. If Y is obtained from X by a single bistellar move of index l then, for $0 \le j \le d$,*

$$g_{j+1}(Y) - g_{j+1}(X) = \begin{cases} +1 & \text{if } j = l \ne d/2 \\ -1 & \text{if } j = d - l \ne d/2 \\ 0 & \text{otherwise.} \end{cases}$$

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Proof. Notice that a bistellar move of index l creates $\binom{d+1-l}{i-l}$ new i-faces, and destroys $\binom{l+1}{d-i+1}$ old i-faces. Thus $f_i(Y) - f_i(X) = \binom{d+1-l}{i-l} - \binom{l+1}{d-i+1}$ for $-1 \leq i \leq d$. Hence the formula (4) for the g-vector yields

$$g_{j+1}(Y) - g_{j+1}(X) = \sum_{i=-1}^{j}(-1)^{j-i}\binom{d-i+1}{j-i}\left[\binom{d+1-l}{i-l} - \binom{l+1}{d-i+1}\right].$$

Now,

$$\sum_{i=-1}^{j}(-1)^{j-i}\binom{d-i+1}{j-i}\binom{l+1}{d-i+1} = \sum_{i=-1}^{j}(-1)^{j-i}\binom{l+1}{d-j+1}\binom{l+j-d}{j-i}$$

$$= \binom{l+1}{d-j+1}\sum_{i=0}^{j+1}(-1)^{i}\binom{l+j-d}{i}$$

$$= (-1)^{j+1}\binom{l+1}{d-j+1}\binom{l+j-d-1}{j+1}.$$

Since $0 \leq j \leq d$ and $0 \leq l \leq d$, this means that

$$\sum_{i=-1}^{j}(-1)^{j-i}\binom{d-i+1}{j-i}\binom{l+1}{d-i+1} = \begin{cases} +1 & \text{if } j = d-l \\ 0 & \text{otherwise.} \end{cases}$$

Replacing l by $d-l$ in this formula, we get

$$\sum_{i=-1}^{j}(-1)^{j-i}\binom{d-i+1}{j-i}\binom{d+1-l}{i-l} = \begin{cases} +1 & \text{if } j = l \\ 0 & \text{otherwise.} \end{cases}$$

Hence the result. $\square$

The following result actually holds for any triangulated $(d+1)$-ball (cf. [33, Corollary 2]). Note that this result implies (6).

Corollary 4.2. *If B is a shellable ball of dimension $d+1$, then $g_j(\partial B) = h_j(B) - h_{d+2-j}(B)$, $0 \leq j \leq d+1$.*

Proof. Induction on the number of facets of B. This is trivial if $B = B_{d+2}^{d+1}$. Else B is obtained from a shellable ball B' with one less facet by a shelling move, say of index l $(0 \leq l \leq d)$. From the definition of the h-vector (for a shellable ball), we have $h_j(B) - h_j(B') = \delta_{j,l+1}$, $h_{d+2-j}(B) - h_{d+2-j}(B') = \delta_{j,d-l+1}$. (Here $\delta_{i,j} = \delta_{ij}$ is Kronecker delta: $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.) Also, ∂B is obtained from $\partial B'$ by a bistellar move of index l (by Lemma 2.3), and hence, by Lemma 4.1, $g_j(\partial B) - g_j(\partial B') = \delta_{j,l+1} - \delta_{j,d-l+1}$. Therefore, we get: $g_j(\partial B) - h_j(B) + h_{d+2-j}(B) = g_j(\partial B') - h_j(B') + h_{d+2-j}(B') = 0$ by induction hypothesis. $\square$

Now, if S is a k-stellated sphere of dimension $d \geq 2k-1$, then, in a sequence of bistellar moves of index $< k$ used to obtain S from S_{d+2}^{d}, the contribution to the g-vector by an l-move is never cancelled by a $(d-l)$-move (and there is no $(d/2)$-move in the sequence). Therefore, as an immediate consequence of Lemma 4.1, we have:

Proposition 4.3. *Let S be a k-stellated sphere of dimension $d \geq 2k-1$. Then any sequence of bistellar moves of index $< k$ used to obtain S from S_{d+2}^d contains exactly $g_{j+1}(S)$ moves of index j $(0 \leq j < k)$. Hence the length of S is given by the formula $l(S) = h_k(S) - 1$.*

Corollary 4.4. *Let S be a k-stellated d-sphere. Then the g-vector of S satisfies $g_j = 0$ for $k+1 \leq j \leq d-k+1$.*

Proof. This is vacuous unless $d \geq 2k$. So, we may assume that $d \geq 2k$. First suppose $k+1 \leq j < \frac{d}{2}+1$. Fix a sequence of bistellar moves of index $< k$ used to obtain S from S_{d+2}^d. Since S is k-stellated and $j > k$, S is also j-stellated. Since $d \geq 2j-1$, Proposition 4.3 implies that this given sequence contains exactly g_j moves of index $j-1$. But, as $j-1 \geq k$, it contains no move of index $j-1$. Thus, $g_j = 0$ for $k+1 \leq j < \frac{d}{2}+1$. If $\frac{d}{2}+1 < j \leq d-k+1$, then $k+1 \leq d+2-j < \frac{d}{2}+1$ and by formula (6), we have $g_j = -g_{d+2-j} = 0$ in this case. Finally, if $j = \frac{d}{2}+1$ then, by formula (6), $g_j = 0$. This completes the proof. □

Remark 4.5. (a) More generally, Corollary 4.4 holds for k-stacked spheres. This may be deduced from Proposition 9.1 in [4] using (4). Therefore, the formula (6) implies that the entire g-vector of a k-stacked sphere of dimension $d \geq 2k$ is determined by the k numbers $g_1, g_2, \ldots, g_k$. (Notice that $g_0 = 1$ for any triangulated sphere.) Equivalently, the f-vector of such a sphere is determined by the k numbers $f_0, f_1, \ldots, f_{k-1}$.

(b) In particular, any k-stacked sphere of dimension $\geq 2k$ has $g_{k+1} = 0$. In [34], McMullen and Walkup posed the famous generalized lower bound conjecture (GLBC): any triangulated sphere of dimension $\geq 2k+1$ satisfies $g_{k+1} \geq 0$ with equality (if and) only if the sphere is k-stacked. Actually, McMullen and Walkup originally posed this conjecture only for polytopal spheres. In this case, the inequality of GLBC was proved by Stanley [39]. Later McMullen gave a simpler proof in [32]. The equality case of GLBC remains open even for polytopal spheres. Notice that, by (6), any triangulated sphere of dimension $d = 2k$ satisfies $g_{k+1} = 0$. So, the hypothesis $d \geq 2k+1$ in the GLBC is essential.

(c) Recall that a simplicial complex X is said to be *l-neighbourly* if any l vertices of X form a face of X, i.e., if $f_{l-1} = \binom{n}{l}$ with $n = f_0(X)$ (equivalently, if $f_i(X) = \binom{n}{i+1}$ for all $i \leq l-1$). Using (4), it is easy to see that if a simplicial complex X of dimension d is l-neighbourly then $g_l(X) = \binom{n+l-d-3}{l}$, where $n = f_0(X)$. If an n-vertex triangulated d-sphere S is l-neighbourly, where $l \geq \frac{d}{2}+1$ and S is not a standard sphere (so that $l \leq n-2$, $n \geq d+3$), then (as $d+2-l \leq l$) S is also $(d+2-l)$-neighbourly, so that we get $g_l + g_{d+2-l} = \binom{n+l-d-3}{l} + \binom{n-l-1}{d+2-l} > 0$, contradicting (6). Thus, if a triangulated d-sphere is not the standard sphere S_{d+2}^d then it can be at most $\lfloor \frac{d+1}{2} \rfloor$-neighbourly. The $\lfloor \frac{d+1}{2} \rfloor$-neighbourly triangulated d-spheres are called the *upper bound spheres* since they attain the componentwise maximum among all the face-vectors of triangulated d-spheres with a given number of vertices.

Corollary 4.6. *Let S be a k-stellated d-sphere which is not the standard d-sphere. Then S is at most k-neighbourly.*

Proof. Suppose S is $(k+1)$-neighbourly. Then, by Remark 4.5 (c) above, $d \geq 2k+1$. Hence, by Corollary 4.4 and Remark 4.5 (c), we get $\binom{n+k-d-2}{k+1} = g_{k+1} = 0$, where $n = f_0(S)$. Hence $n+k-d-2 < k+1$, i.e., $n < d+3$. Hence $n = d+2$ and therefore $S = S_{d+2}^d$. □

The following lemma is perhaps well known. For a more conceptual proof in the case of polytopal spheres, see [33, Theorem 5.1].

Lemma 4.7. *If X is a simplicial complex of dimension d, then*

$$\sum_{x \in V(X)} g_j(\mathrm{lk}_X(x)) = (d+2-j)g_j(X) + (j+1)g_{j+1}(X) \quad for \ \ 0 \leq j \leq d.$$

Proof. A simple two-way counting yields $\sum_{x \in V(X)} f_i(\mathrm{lk}_X(x)) = (i+2)f_{i+1}(X)$. Therefore, we get :

$$
\begin{aligned}
\sum_{x \in V(X)} g_j(\mathrm{lk}_X(x)) &= \sum_{i=-1}^{j-1} (-1)^{j-i-1}\binom{d-i}{j-i-1} \sum_{x \in V(X)} f_i(\mathrm{lk}_X(x)) \\
&= \sum_{i=-1}^{j-1} (-1)^{j-i-1}\binom{d-i}{j-i-1}(i+2)f_{i+1}(X) \\
&= \sum_{i=0}^{j} (-1)^{j-i}\binom{d-i+1}{j-i}(i+1)f_i(X) \\
&= \sum_{i=-1}^{j} (-1)^{j-i}\binom{d-i+1}{j-i}(i+1)f_i(X) \\
&= \sum_{i=-1}^{j} (-1)^{j-i}\left[(j+1)\binom{d-i+1}{j-i} - (d+2-j)\binom{d-i+1}{j-i-1}\right]f_i(X) \\
&= (d+2-j)\sum_{i=-1}^{j-1} (-1)^{j-i-1}\binom{d-i+1}{j-i-1}f_i(X) \\
&\quad + (j+1)\sum_{i=-1}^{j} (-1)^{j-i}\binom{d-i+1}{j-i}f_i(X) \\
&= (d+2-j)g_j(X) + (j+1)g_{j+1}(X).
\end{aligned}
$$

$\square$

Proposition 4.8. *The g-vector of any member M of $\mathcal{W}_k(d)$ satisfies*

$$\frac{g_j}{\binom{d+2}{j}} = (-1)^{j-k-1}\frac{g_{k+1}}{\binom{d+2}{k+1}} \quad for \ \ k+1 < j \leq d-k+1.$$

If, further, $d \geq 2k$ and d is even, then the Euler characteristic χ of M is given by the formula

$$(-1)^k(\chi-2) = \frac{2\,g_{k+1}}{\binom{d+2}{k+1}}.$$

Proof. The link of any vertex x in M is a k-stellated sphere of dimension $d-1$. Therefore, by Corollary 4.4, we have $g_j(\mathrm{lk}_M(x)) = 0$ for $k+1 \leq j \leq d-k$. Hence Lemma 4.7 yields $(d+2-j)g_j + (j+1)g_{j+1} = 0$, i.e.,

$$\frac{g_{j+1}}{\binom{d+2}{j+1}} = -\frac{g_j}{\binom{d+2}{j}} \quad for \ \ k+1 \leq j \leq d-k.$$

Hence, by finite induction on j, we get the formula

$$\frac{g_j}{\binom{d+2}{j}} = (-1)^{j-k-1}\frac{g_{k+1}}{\binom{d+2}{k+1}} \quad for \ \ k+1 \leq j \leq d-k+1.$$

Now, suppose $d \geq 2k$ is even. Then $k + 1 \leq \frac{d}{2} + 1 \leq d - k + 1$. Hence we have

$$\frac{g_{\frac{d}{2}+1}}{\binom{d+2}{\frac{d}{2}+1}} = (-1)^{\frac{d}{2}-k} \frac{g_{k+1}}{\binom{d+2}{k+1}}.$$

On the other hand, (7) contains the formula

$$\frac{2\, g_{\frac{d}{2}+1}}{\binom{d+2}{\frac{d}{2}+1}} = (-1)^{\frac{d}{2}}(\chi - 2).$$

Comparing these two, we get the formula for χ. $\qquad\qquad\square$

Corollary 4.9. *Let $d \geq 2k$ and $M \in \mathcal{W}_k(d)$. Then the entire g-vector of M is determined by the $k + 1$ numbers g_i, $1 \leq i \leq k + 1$. Equivalently, the face-vector of M is determined by the $k + 1$ numbers f_i, $0 \leq i \leq k$.*

Proof. Note that $g_0 = 1$. Also, Proposition 4.8 determines $g_{k+2}, \ldots, g_{d-k+1}$ in terms of g_{k+1}. Klee's formula (7) determines $g_{d-k+2}, \ldots, g_{d+1}$ in terms of $g_1, \ldots, g_k$ and the Euler characteristic χ of M. But $\chi = 0$ if d is odd, and Proposition 4.8 determines χ in terms of g_{k+1} when d is even. $\qquad\qquad\square$

Notation: For any simplicial complex X with vertex set $V(X)$ and any set A, $X[A]$ will denote the induced subcomplex of X on the vertex set $A \cap V(X)$. Thus, $X[A] = \{\alpha \in X : \alpha \subseteq A\}$.

Definition 4.10. Let $X = X_m^d$ be a simplicial complex of dimension d on m vertices. Let $\mathbb{F}$ be a field. Then we define the beta-, sigma-, and mu-vector of X (with respect to $\mathbb{F}$) as follows. The *beta-vector* of X is the vector $(\beta_0, \beta_1, \ldots, \beta_d)$, where $\beta_i = \beta_i(X) = \beta_i(X; \mathbb{F})$ is the ith Betti number of X with $\mathbb{F}$-coefficients. That is, $\beta_i(X) = \dim_{\mathbb{F}} H_i(X; \mathbb{F})$, $0 \leq i \leq d$. More generally, if Y is a subcomplex of X, we use $\beta_i(X, Y)$ to denote $\dim_{\mathbb{F}} H_i(X, Y; \mathbb{F})$, $0 \leq i \leq d$. As usual, $\widetilde{\beta}_i$ will denote the corresponding reduced Betti numbers. Thus $\widetilde{\beta}_i = \beta_i$ if $i \neq 0$ and $\widetilde{\beta}_0 = \beta_0 - 1$.

The *sigma-vector* $(\sigma_0, \sigma_1, \ldots, \sigma_d)$ of X (with respect to $\mathbb{F}$) is defined by

$$\sigma_i = \sigma_i(X; \mathbb{F}) = \sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \sum_{A \in \binom{V(X)}{j}} \widetilde{\beta}_i(X[A]), \quad 0 \leq i \leq d.$$

We define the *mu-vector* $(\mu_0, \ldots, \mu_d)$ of X (with respect to $\mathbb{F}$) by:

$$\mu_0 = \mu_0(X; \mathbb{F}) = 1$$
$$\mu_i = \mu_i(X; \mathbb{F}) = \delta_{i1} + \frac{1}{m} \sum_{x \in V(X)} \sigma_{i-1}(\mathrm{lk}_X(x)), \quad 1 \leq i \leq d.$$

(By a slight abuse of notation, let's write $\emptyset$ for the trivial simplicial complex whose only face is the empty set. Here, we have adopted the convention $\widetilde{\beta}_0(\emptyset) = -1$ and $\widetilde{\beta}_i(\emptyset) = 0$ if $i \neq 0$. This convention accounts for the Kronecker delta in the definition of the mu-vector.)

Notation: We write $\mathrel{-\!\!\circ}$ for the covering relation for set inclusion. Thus, for sets A and B, $A \mathrel{-\!\!\circ} B$ means that $A \subseteq B$ and $\#(B \setminus A) = 1$.

With this notation, we have:

Lemma 4.11. *Let X be a 2-neighbourly simplicial complex of dimension d on m vertices. Then, the mu-vector of X (with respect to any field) is given by:*

$$\mu_i = \frac{1}{m} \sum_{j=1}^{m} \frac{1}{\binom{m-1}{j-1}} \sum_{\substack{A, B \subseteq V(X), \\ \#(B) = j, \\ A \multimap B}} \beta_i(X[B], X[A]), \quad 0 \le i \le d.$$

Proof. Since X is 2-neighbourly, so is any induced subcomplex of X. Hence, for $A \multimap B \subseteq V(X)$, $\beta_0(X[B], X[A]) = 1$ if $\#(B) = 1$, $A = \emptyset$ and $\beta_0(X[B], X[A]) = 0$ otherwise. Hence the formula holds for $i = 0$. So, let $1 \le i \le d$. For any $x \in V(X)$, let L_x be the link of x in X. Then, each L_x is a simplicial complex of dimension $d - 1$ with exactly $m - 1$ vertices. Let $V = V(X)$ be the vertex set of X. Thus, the vertex set of L_x is $V \setminus \{x\}$. We have

$$\sigma_{i-1}(L_x) = \sum_{j=1}^{m} \frac{1}{\binom{m-1}{j-1}} \sum_{A \in \binom{V \setminus \{x\}}{j-1}} \widetilde{\beta}_{i-1}(L_x[A]).$$

But the exact homology sequence for pairs and the excision theorem yield

$$\widetilde{\beta}_{i-1}(L_x[A]) = \begin{cases} \beta_i(x * L_x[A], L_x[A]) = \beta_i(X[A \sqcup \{x\}], X[A]) & \text{when } A \ne \emptyset \text{ and} \\ \beta_i(X[A \sqcup \{x\}], X[A]) - \delta_{i1} & \text{when } A = \emptyset. \end{cases}$$

Therefore, from the definition of the mu-vector of X, we have

$$\mu_i = \frac{1}{m} \sum_{x \in V} \sum_{j=1}^{m} \frac{1}{\binom{m-1}{j-1}} \sum_{A \in \binom{V \setminus \{x\}}{j-1}} \beta_i(X[A \sqcup \{x\}], X[A]).$$

$\square$

The following linear algebra lemma must be well known. But, we could not find a reference to it in the required form.

Lemma 4.12. *Let $V_1 \xrightarrow{T_1} V_2 \to \cdots \to V_{2m} \xrightarrow{T_{2m}} V_{2m+1}$ be an exact sequence of linear transformations between finite dimensional vector spaces (involving an even number $2m$ of arrows). Then $\sum_{i=1}^{2m+1} (-1)^i \dim(V_i) \le 0$. Equality holds here if and only if T_1 is injective and T_{2m} is surjective.*

Proof. From the assumed exactness, we have, for $1 < i < 2m + 1$, $\dim(V_i) = \operatorname{rank}(T_i) + \operatorname{nullity}(T_i) = \operatorname{rank}(T_i) + \operatorname{rank}(T_{i-1})$. Therefore,

$$\begin{aligned} \sum_{i=1}^{2m+1} (-1)^i \dim(V_i) &= -\dim(V_1) - \dim(V_{2m+1}) + \sum_{i=2}^{2m} (-1)^i \left[\operatorname{rank}(T_i) + \operatorname{rank}(T_{i-1}) \right] \\ &= -[\dim(V_1) - \operatorname{rank}(T_1)] - [\dim(V_{2m+1}) - \operatorname{rank}(T_{2m})] \\ &\le 0 + 0 = 0. \end{aligned}$$

From this argument, it is immediate that the necessary and sufficient condition for equality here is that T_1 should be injective and T_{2m} surjective. $\square$

The following proposition is our version of the usual combinatorial Morse theory. In particular, the parts (a) and (b) of this proposition are the strong and weak Morse inequalities

(averaged over all possible "regular simplexwise linear" functions; compare [25]). For an alternative combinatorial version of Morse theory, consult [18]. As we shall see, the version developed here is specially suited to the study of $\mathbb{F}$-tightness of $\mathbb{F}$-orientable 2-neighbourly triangulated closed manifolds.

Proposition 4.13. *Let X be a 2-neighbourly simplicial complex of dimension d. Then the mu- and beta-vectors of X are related as follows (with respect to any given field $\mathbb{F}$).*

(a) *For $0 \leq j \leq d$, $\sum_{i=0}^{j}(-1)^{j-i}\mu_i \geq \sum_{i=0}^{j}(-1)^{j-i}\beta_i$, with equality for $j = d$.*

(b) *For $0 \leq j \leq d$, $\mu_j \geq \beta_j$.*

(c) *The following are equivalent for any fixed index j $(0 \leq j \leq d)$:*

   (i) *$\sum_{i=0}^{j}(-1)^{j-i}\mu_i = \sum_{i=0}^{j}(-1)^{j-i}\beta_i$, and*
   (ii) *for any induced subcomplex Y of X, the morphism $H_j(Y;\mathbb{F}) \to H_j(X;\mathbb{F})$ induced by the inclusion map $Y \hookrightarrow X$ is injective.*

(d) *The following are equivalent for any fixed index j $(0 \leq j \leq d)$:*

   (iii) *$\mu_j = \beta_j$, and*

   (iv) *for any induced subcomplex Y of X, both the morphisms $H_{j-1}(Y;\mathbb{F}) \to H_{j-1}(X;\mathbb{F})$ and $H_j(Y;\mathbb{F}) \to H_j(X;\mathbb{F})$ induced by the inclusion map $Y \hookrightarrow X$ are injective.*

(e) *If, further, X is an $\mathbb{F}$-orientable closed manifold, then $\beta_{d-j} = \beta_j$ and $\mu_{d-j} = \mu_j$ for $0 \leq j \leq d$.*

Proof. (a) Fix an index j and subsets $A \subseteq B$ of $V(X)$. We have the following exact sequence of relative homology: $H_j(X[A]) \to H_j(X[B]) \to H_j(X[B], X[A]) \to H_{j-1}(X[A]) \to \cdots \to H_0(X[A]) \to H_0(X[B]) \to H_0(X[B], X[A]) \to 0$. If necessary, we may append an extra $0 \to 0$ at the extreme right, to ensure that this exact sequence has an even number of arrows. Applying Lemma 4.12 to this sequence, we get:

$$\sum_{i=0}^{j}(-1)^{j-i}\beta_i(X[B], X[A]) \geq \sum_{i=0}^{j}(-1)^{j-i}\left(\beta_i(X[B]) - \beta_i(X[A])\right) \tag{8}$$

for all pairs $A \subseteq B$ of subsets of $V(X)$.

Since the extreme right arrow in the above sequence is trivially a surjection, Lemma 4.12 says that, for any given pair $A \subseteq B$, equality in (8) holds if and only if the morphism $H_j(X[A]) \to H_j(X[B])$ induced by the inclusion map $A \hookrightarrow B$ is an injection.

Now, in view of Lemma 4.11, taking the appropriate weighted sum of the inequalities (8) over all pairs (A, B) with $A \multimap B$, we get

$$\sum_{i=0}^{j}(-1)^{j-i}\mu_i \geq \frac{1}{m}\sum_{l=1}^{m}\frac{1}{\binom{m-1}{l-1}}\sum_{\substack{A, B \subseteq V(X), \\ \#(B) = l, \\ A \multimap B}}\sum_{i=0}^{j}(-1)^{j-i}\left(\beta_i(X[B]) - \beta_i(X[A])\right). \tag{9}$$

Here $m = \#(V(X))$.

Equality holds in (9) if and only if $H_j(X[A]) \to H_j(X[B])$ is injective for all pairs (A, B) with $A \multimap B \subseteq V(X)$. In particular, since X is d-dimensional and each d-cycle of $X[A]$ is a d-cycle of $X[B]$, equality holds in (9) for $j = d$. The right hand side of (9) may be written

as $\sum_{i=0}^{j}(-1)^{j-i}\sum_{C\subseteq V(X)}\alpha(C)\beta_i(X[C])$, where the coefficients $\alpha(C)$ are given in terms of $n := \#(C)$ by the formula

$$\alpha(C) = \frac{1}{m}\frac{1}{\binom{m-1}{n-1}}\sum_{\substack{A\subseteq V(X),\\ A\longrightarrow\!\!\circ\, C}}1 - \frac{1}{m}\frac{1}{\binom{m-1}{n}}\sum_{\substack{B\subseteq V(X),\\ C\longrightarrow\!\!\circ\, B}}1 = \frac{1}{m}\left(\frac{n}{\binom{m-1}{n-1}} - \frac{m-n}{\binom{m-1}{n}}\right),$$

where the first term occurs only for $n > 0$, and the second term occurs only for $n < m$. This simplifies to

$$\alpha(C) = \begin{cases} +1 & \text{if } C = V(X) \\ -1 & \text{if } C = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the right hand side of (9) simplifies to $\sum_{i=0}^{j}(-1)^{j-i}(\beta_i(X) - \beta_i(\emptyset)) = \sum_{i=0}^{j}(-1)^{j-i}\beta_i$.

(b) We have,

$$\mu_j = \sum_{i=0}^{j-1}(-1)^{j-1-i}\mu_i + \sum_{i=0}^{j}(-1)^{j-i}\mu_i \geq \sum_{i=0}^{j-1}(-1)^{j-1-i}\beta_i + \sum_{i=0}^{j}(-1)^{j-i}\beta_i = \beta_j.$$

(c) By the proof of part (a), we see that the equality (i) holds if and only if $H_j(X[A]) \to H_j(X[B])$ is injective for all pairs $A \longrightarrow\!\!\circ\, B \subseteq V(X)$. Now, let Y be an induced subcomplex of X, say with vertex set A. Take a sequence $A = A_1 \longrightarrow\!\!\circ\, A_2 \longrightarrow\!\!\circ\, \cdots \longrightarrow\!\!\circ\, A_n = V(X)$. Then the morphism $H_j(Y) \to H_j(X)$ is the composition of the morphisms $H_j(X[A_1]) \to H_j(X[A_2]) \to \cdots \to H_j(X[A_n])$. If (i) holds then $H_j(Y) \to H_j(X)$, being a composition of injective morphisms, is itself injective. Thus, (ii) holds. Conversely, if (ii) holds, then for any pair $A_1 \longrightarrow\!\!\circ\, A_2 \subseteq V(X)$, choose $Y = X[A_1]$. Then the composition of the above sequence of morphisms is injective. So, the first morphism $H_j(X[A_1]) \to H_j(X[A_2])$ in this sequence must be injective. Therefore (i) holds.

(d) From the proof of part (b), we see that (iii) holds if and only if $\sum_{i=0}^{j-1}(-1)^{j-1-i}\mu_i = \sum_{i=0}^{j-1}(-1)^{j-1-i}\beta_i$ and $\sum_{i=0}^{j}(-1)^{j-i}\mu_i = \sum_{i=0}^{j}(-1)^{j-i}\beta_i$. Therefore, part (d) follows from part (c).

(e) In this case, Alexander duality yields $\beta_{d-i} = \beta_i$. For any $A \subseteq V(X)$, let A^c denote the complement of A with respect to $V(X)$. Observe that $(A, B) \mapsto (B^c, A^c)$ is a permutation of the set of all pairs (A, B) with $A \longrightarrow\!\!\circ\, B \subseteq V(X)$. Also, by Alexander duality, we have $\beta_{d-i}(X[B], X[A]) = \beta_i(X[A^c], X[B^c])$. Therefore, Lemma 4.11 yields $\mu_{d-i} = \mu_i$. $\qquad\square$

Now we recall:

Definition 4.14. Let X be a d-dimensional simplicial complex and $\mathbb{F}$ be a field. We say that X is *tight with respect to* $\mathbb{F}$ (or, in short, $\mathbb{F}$-*tight*) if (i) X is connected, and (ii) for all induced subcomplexes Y of X and for all $0 \leq j \leq d$, the morphism $H_j(Y; \mathbb{F}) \to H_j(X; \mathbb{F})$ induced by the inclusion map $Y \hookrightarrow X$ is injective.

Note that, for fields $\mathbb{F}_1 \subseteq \mathbb{F}_2$, X is $\mathbb{F}_1$-tight if and only if X is $\mathbb{F}_2$-tight. Therefore, in studying $\mathbb{F}$-tightness, we may, without loss of generality, restrict to prime fields $\mathbb{F}$, i.e., $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{Z}_p$, p prime. Moreover, for any simplicial complex X, the following are equivalent: (a) X is $\mathbb{F}$-tight for all fields $\mathbb{F}$, (b) X is $\mathbb{Z}_p$-tight for all primes p, and (c) X is $\mathbb{Q}$-tight. In view of this observation, we shall say that X is *tight* if it is $\mathbb{Q}$-tight.

Clearly, if X is $\mathbb{F}$-tight then so is every connected induced subcomplex of X. From this trivial observation, it is easy to see that the standard sphere S^d_{d+2} is the only tight triangulated d-sphere, and the standard ball B^d_{d+1} is the only tight triangulated d-ball. We also have:

Proposition 4.15. *Let X be an $\mathbb{F}$-tight simplicial complex (for some field $\mathbb{F}$).*

(a) *If X is $(k-1)$-connected in the sense of homotopy (for some $k \geq 1$) then X is $(k+1)$-neighbourly.*

(b) *If X is a triangulated closed manifold, then X is $\mathbb{F}$-orientable.*

Proof. (a) Suppose not. Let l be the smallest integer such that X is not $(l+1)$-neighbourly. We have $1 \leq l \leq k$. The induced subcomplex $X[\alpha]$ of X on any missing l-face α is an S^{l-1}_{l+1}. Since $H_{l-1}(X[\alpha]; \mathbb{F}) \to H_{l-1}(X; \mathbb{F})$ is injective, it follows that $\widetilde{H}_{l-1}(X; \mathbb{F}) \neq 0$. This is a contradiction since $l \leq k$ and X is $(k-1)$-connected.

(b) This is trivial if $\dim(X) = 1$. So, assume $d := \dim(X) \geq 2$. Fix a vertex x of X, and let L be the link of x in X. Then, L is a homology $(d-1)$-sphere. In particular, $\beta_{d-1}(L) = 1 > 0$. Thus, all the terms in the sum defining $\sigma_{d-1}(L)$ (cf. Definition 4.10) are non-negative, and at least one is positive. Hence $\sigma_{d-1}(L) > 0$ for any vertex link L of X. Thus, all the terms in the sum defining $\mu_d(X)$ are > 0. Therefore, the mu-vector of X satisfies $\mu_d > 0$. On the other hand, if X is not $\mathbb{F}$-orientable, then $\beta_d = 0$. Thus $\mu_d \neq \beta_d$. Also, by part (a), X is 2-neighbourly. Therefore, by Proposition 4.13 (d), there is an induced subcomplex Y of X such that $H_{d-1}(Y; \mathbb{F}) \to H_{d-1}(X; \mathbb{F})$ is not injective (note that $H_d(Y; \mathbb{F}) \to H_d(X; \mathbb{F})$ is always injective). This contradicts the $\mathbb{F}$-tightness of X. $\square$

Another way of stating Proposition 4.15 (b) is that, if X is a triangulated closed manifold which is not orientable (over $\mathbb{Z}$), then $\mathbb{F} = \mathbb{Z}_2$ is essentially the only choice for a field for which X has a chance of being $\mathbb{F}$-tight. Note that, as a special case ($k = 1$) of Proposition 4.15 (a), any $\mathbb{F}$-tight simplicial complex is necessarily 2-neighbourly (whatever the field $\mathbb{F}$).

Now we have:

Proposition 4.16. *A simplicial complex X of dimension d is $\mathbb{F}$-tight if and only if X is 2-neighbourly and $\mu_i(X; \mathbb{F}) = \beta_i(X; \mathbb{F})$ for all indices i, $0 \leq i \leq d$.*

Proof. This is immediate from Proposition 4.13 (d) and Proposition 4.15 (a). $\square$

5 A tightness criterion for members of $\mathcal{W}_k(d)$

In the following result, B^e stands for an arbitrary triangulated ball of dimension e, and S^{e-1} stands for an arbitrary triangulated sphere of dimension $e - 1$. Here $e \geq 0$, and $S^{-1} = \emptyset$.

Lemma 5.1. *Let X, Y be simplicial complexes such that $Y = X \cup B^e$. Suppose $X \cap B^e = B^{e-1}$ (with $e \geq 1$) or $X \cap B^e = S^{e-1}$ (with $e \geq 0$). Then the reduced Betti numbers (with respect to any field $\mathbb{F}$) of X and Y are related as follows.*

(a) *If $X \cap B^e = B^{e-1}$ then $\widetilde{\beta}_i(Y) = \widetilde{\beta}_i(X)$ for all i.*

(b) *If $X \cap B^e = S^{e-1}$ then*

$$\text{either} \quad \widetilde{\beta}_i(Y) - \widetilde{\beta}_i(X) = \begin{cases} +1 & if \quad i = e \\ 0 & if \quad i \neq e, \end{cases}$$

$$\text{or} \quad \widetilde{\beta}_i(Y) - \widetilde{\beta}_i(X) = \begin{cases} -1 & if \quad i = e - 1 \\ 0 & if \quad i \neq e - 1. \end{cases}$$

Proof. When $e \neq 0$, this is immediate from Mayer-Vietoris theorem for reduced (simplicial) homology. When $e = 0$, the hypothesis says that Y is the disjoint union of X and a point, so that the result is trivial in this case (and the first alternative holds). $\square$

Lemma 5.2. *Let X, Y be simplicial complexes of dimension d such that Y is obtained from X by a single bistellar move $\alpha \mapsto \beta$, say of index t $(0 \leq t \leq d)$. Then, for any subset A of $V(Y)$, the reduced Betti numbers (with respect to any field $\mathbb{F}$) of $X[A]$ and $Y[A]$ are related as follows.*

(i) If $A \supseteq \beta$, $A \cap \alpha = \emptyset$ then

$$\text{either} \quad \widetilde{\beta}_i(Y[A]) - \widetilde{\beta}_i(X[A]) = \begin{cases} +1 & if \quad i = t \\ 0 & if \quad i \neq t, \end{cases}$$

$$\text{or} \quad \widetilde{\beta}_i(Y[A]) - \widetilde{\beta}_i(X[A]) = \begin{cases} -1 & if \quad i = t - 1 \\ 0 & if \quad i \neq t - 1. \end{cases}$$

(ii) If $A \supseteq \alpha$, $A \cap \beta = \emptyset$ then

$$\text{either} \quad \widetilde{\beta}_i(Y[A]) - \widetilde{\beta}_i(X[A]) = \begin{cases} -1 & if \quad i = d - t \\ 0 & if \quad i \neq d - t, \end{cases}$$

$$\text{or} \quad \widetilde{\beta}_i(Y[A]) - \widetilde{\beta}_i(X[A]) = \begin{cases} +1 & if \quad i = d - t - 1 \\ 0 & if \quad i \neq d - t - 1. \end{cases}$$

(iii) In all other cases, $\widetilde{\beta}_i(Y[A]) = \widetilde{\beta}_i(X[A])$ for all i.

Proof. If $A \supseteq \beta$, $A \cap \alpha = \emptyset$ then we have $Y[A] = X[A] \cup \overline{\beta}$ and $X[A] \cap \overline{\beta} = \partial\beta$. If $A \supseteq \alpha$, $A \cap \beta = \emptyset$ then we have $X[A] = Y[A] \cup \overline{\alpha}$ and $Y[A] \cap \overline{\alpha} = \partial\alpha$. So, the result is immediate from Lemma 5.1 in these cases. (Remember that $\dim(\beta) = t$ and $\dim(\alpha) = d - t$.)

If $A \supseteq \alpha \cup \beta$ then $Y[A]$ is obtained from $X[A]$ by the bistellar move $\alpha \mapsto \beta$. Hence $Y[A]$ and $X[A]$ are homeomorphic in this case, and the result follows.

If A contains neither α nor β, then $Y[A] = X[A]$, and the result is trivial.

If $A \supseteq \beta$ and $\alpha_0 := A \cap \alpha$ is a proper non-empty subset of α, then $Y[A] = X[A] \cup B^e$ and $X[A] \cap B^e = B^{e-1}$, where $e = t + \#(\alpha_0) > 0$, $B^e = \overline{\alpha_0 \cup \beta}$, $B^{e-1} = \overline{\alpha_0} * \partial\beta$. Hence the result follows from Lemma 5.1.

If $A \supseteq \alpha$ and $\beta_0 := A \cap \beta$ is a proper non-empty subset of β, then $X[A] = Y[A] \cup B^e$ and $Y[A] \cap B^e = B^{e-1}$, where $e = d - t + \#(\beta_0) > 0$, $B^e = \overline{\alpha \cup \beta_0}$, $B^{e-1} = \overline{\beta_0} * \partial\alpha$. Hence the result follows from Lemma 5.1 in this case also. $\square$

Now, we are in a position to prove a crucial result on the sigma-vectors of k-stellated spheres:

Proposition 5.3. *For $k \geq 1$, let S be an m-vertex k-stellated triangulated sphere of dimension $d \geq 2k - 1$. Then, with respect to any field, the sigma-vector of S is related to its g-vector by:*

(a) $\sigma_i = 0$ *for* $k \leq i \leq d - k - 1$,

(b) $\displaystyle\sum_{i=0}^{l}(-1)^{l-i}\sigma_i \leq \frac{m+1}{d+3}\sum_{i=0}^{l+1}(-1)^{l+1-i}\frac{g_i}{\binom{d+2}{i}}$ *for* $0 \leq l \leq k-2$, *and*

(c) $\displaystyle\sum_{i=0}^{l}(-1)^{l-i}\sigma_i = \frac{m+1}{d+3}\sum_{i=0}^{l+1}(-1)^{l+1-i}\frac{g_i}{\binom{d+2}{i}}$ *for* $k-1 \leq l \leq d-k-1$.

Proof. Induction on the length $l(S)$ of S. If $l(S) = 0$, then $S = S_{d+2}^{d}$. In this case, $\sigma_i(S) = -\delta_{i0}$ for $0 \leq i < d$, $g_i(S) = \delta_{i0}$ for $0 \leq i \leq d+1$, and $m = d+2$. So, the result is trivial in this case. Now, assume $l(S) > 0$. Then S is obtained from a shorter k-stellated d-sphere S' by a single bistellar move $\alpha \mapsto \beta$, say of index t $(0 \leq t < k)$.

For $k \leq i \leq d-k-1$, we have $t < i < d-t-1$ and hence by Lemma 5.2, $\widetilde{\beta}_i(S[A]) = \widetilde{\beta}_i(S'[A])$ for all subsets A of $V(S)$. Taking an appropriate weighted sum of these equalities over all sets A, we get $\sigma_i(S) = \sigma_i(S') = 0$ for $k \leq i \leq d-k-1$. Here the last equality is by induction hypothesis. This proves part (a).

Fix an index l such that $0 \leq l \leq d-k-1$ and let $0 \leq i \leq l$. Then $i \leq d-k-1 < d-t-1$, and hence by Lemma 5.2, $\widetilde{\beta}_i(S[A]) = \widetilde{\beta}_i(S'[A])$ unless $A \supseteq \beta$, $A \cap \alpha = \emptyset$. So, let $\mathcal{A}$ be the collection of all subsets A of $V(S)$ such that $A \supseteq \beta$ and $A \cap \alpha = \emptyset$. For $A \in \mathcal{A}$, we have $\widetilde{\beta}_i(S[A]) = \widetilde{\beta}_i(S'[A])$ for $i \neq t, t-1$, and either $\widetilde{\beta}_t(S[A]) = \widetilde{\beta}_t(S'[A]) + 1$, $\widetilde{\beta}_{t-1}(S[A]) = \widetilde{\beta}_{t-1}(S'[A])$, or else $\widetilde{\beta}_t(S[A]) = \widetilde{\beta}_t(S'[A])$, $\widetilde{\beta}_{t-1}(S[A]) = \widetilde{\beta}_{t-1}(S'[A]) - 1$. Let $\mathcal{A}^+$ be the set of all $A \in \mathcal{A}$ for which the first alternative holds. Then we get:

$$\sum_{i=0}^{l}(-1)^{l-i}\left[\widetilde{\beta}_i(S[A]) - \widetilde{\beta}_i(S'[A])\right] = \begin{cases} 0 & \text{if } A \notin \mathcal{A} \text{ or if } l < t-1 \\ 0 & \text{if } A \in \mathcal{A}^+ \text{ and } l = t-1 \\ (-1)^{l-t} & \text{otherwise.} \end{cases} \quad (10)$$

First consider the case $l \leq t-1$ (which can occur only for $l < k-1$ and $t > 0$). Then (10) implies

$$\sum_{i=0}^{l}(-1)^{l-i}\widetilde{\beta}_i(S[A]) \leq \sum_{i=0}^{l}(-1)^{l-i}\widetilde{\beta}_i(S'[A]) \text{ for all } A \subseteq V(S).$$

Taking the appropriate weighted sum of these inequalities over all A, we get

$$\sum_{i=0}^{l}(-1)^{l-i}\sigma_i(S) \leq \sum_{i=0}^{l}(-1)^{l-i}\sigma_i(S') \leq \frac{m+1}{d+3}\sum_{i=0}^{l+1}(-1)^{l+1-i}\frac{g_i(S')}{\binom{d+2}{i}}$$

$$= \frac{m+1}{d+3}\sum_{i=0}^{l+1}(-1)^{l+1-i}\frac{g_i(S)}{\binom{d+2}{i}},$$

where the second inequality is by induction hypothesis and the final equality holds since by Lemma 4.1, we have $g_i(S) = g_i(S')$ for $i \leq l+1 \leq t$. This completes the induction step in this case.

Next consider the case $0 < t \leq l \leq d-k-1$. In this case, (10) says:

$$\sum_{i=0}^{l}(-1)^{l-i}\left[\widetilde{\beta}_i(S[A]) - \widetilde{\beta}_i(S'[A])\right] = \begin{cases} (-1)^{l-t} & \text{if } A \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}$$

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Notice that there are exactly $\binom{m-d-2}{j-t-1}$ j-sets in $\mathcal{A}$. Therefore, adding these equations over all j-subsets of $V := V(S)$ we get (for $0 \le j \le m$):

$$\sum_{i=0}^{l} (-1)^{l-i} \sum_{A \in \binom{V}{j}} \left[\tilde{\beta}_i(S[A]) - \tilde{\beta}_i(S'[A]) \right] = (-1)^{l-t} \binom{m-d-2}{j-t-1}. \tag{11}$$

Dividing this equation by $\binom{m}{j}$ and adding over all j, we get:

$$\sum_{i=0}^{l} (-1)^{l-i} \left[\sigma_i(S) - \sigma_i(S') \right] = (-1)^{l-t} \sum_{j=0}^{m} \frac{\binom{m-d-2}{j-t-1}}{\binom{m}{j}}.$$

But, we have the computation

$$
\begin{aligned}
\sum_{j=0}^{m} \frac{\binom{m-d-2}{j-t-1}}{\binom{m}{j}} &= \sum_{j=t+1}^{m+t-d-1} \frac{\binom{m-d-2}{j-t-1}}{\binom{m}{j}} = \sum_{j=0}^{m-d-2} \frac{\binom{m-d-2}{j}}{\binom{m}{j+t+1}} \\
&= (m+1) \sum_{j=0}^{m-d-2} \binom{m-d-2}{j} \int_0^1 x^{j+t+1}(1-x)^{m-j-t-1} dx \\
&= (m+1) \int_0^1 x^{t+1}(1-x)^{d+1-t} dx = \frac{m+1}{d+3} \frac{1}{\binom{d+2}{t+1}},
\end{aligned}
$$

where we have made two uses of Euler's famous identity:

$$\int_0^1 x^a (1-x)^b dx = \frac{1}{(a+b+1)\binom{a+b}{a}},$$

for non-negative integers a, b.

So, we have

$$\sum_{j=0}^{m} \frac{\binom{m-d-2}{j-t-1}}{\binom{m}{j}} = \frac{m+1}{d+3} \frac{1}{\binom{d+2}{t+1}}. \tag{12}$$

Thus we get:

$$\sum_{i=0}^{l} (-1)^{l-i} \left[\sigma_i(S) - \sigma_i(S') \right] = (-1)^{l-t} \frac{m+1}{d+3} \frac{1}{\binom{d+2}{t+1}}.$$

Therefore, induction hypothesis gives the following inequality for $0 < t \le l \le d-k-1$ (with equality for $k-1 \le l \le d-k-1$):

$$
\begin{aligned}
\sum_{i=0}^{l} (-1)^{l-i} \sigma_i(S) &\le (-1)^{l-t} \frac{m+1}{d+3} \frac{1}{\binom{d+2}{t+1}} + \frac{m+1}{d+3} \sum_{i=0}^{l+1} (-1)^{l+1-i} \frac{g_i(S')}{\binom{d+2}{i}} \\
&= \frac{m+1}{d+3} \sum_{i=0}^{l+1} (-1)^{l+1-i} \frac{g_i(S') + \delta_{i,t+1}}{\binom{d+2}{i}} \\
&= \frac{m+1}{d+3} \sum_{i=0}^{l+1} (-1)^{l+1-i} \frac{g_i(S)}{\binom{d+2}{i}},
\end{aligned}
$$

where the last equality holds since by Lemma 4.1, we have $g_i(S) = g_i(S') + \delta_{i,t+1}$ as $d \geq 2k-1$ (so that $d \neq 2t$) and $i - 1 \leq l \leq d - k < d - t$. This completes the induction step in the second case.

Finally, consider the case $0 = t \leq l \leq k - d - 1$. In this case, β is a vertex of S not in S'. Let $V' = V \setminus \{\beta\}$ be the vertex set of S'. (Thus, S' has $m - 1$ vertices in this case.) Dividing equation (11) (with $t = 0$) by $\binom{m}{j}$ and adding over all j $(0 \leq j \leq m)$ we get (in view of (12))

$$\sum_{i=0}^{l} (-1)^{l-i} \sigma_i(S) = (-1)^l \frac{m+1}{(d+3)(d+2)} + \sum_{i=0}^{l} (-1)^{l-i} \sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \sum_{A \in \binom{V}{j}} \widetilde{\beta}_i(S'[A]).$$

Now, for each $i \leq l$,

$$\sum_{j=0}^{m} \frac{1}{\binom{m}{j}} \sum_{A \in \binom{V}{j}} \widetilde{\beta}_i(S'[A]) = \sum_{j=0}^{m-1} \frac{1}{\binom{m}{j}} \sum_{A \in \binom{V}{j}} \widetilde{\beta}_i(S'[A]) = \sum_{j=0}^{m-1} \frac{1}{\binom{m}{j}} \sum_{B \in \binom{V'}{j} \sqcup \binom{V'}{j-1}} \widetilde{\beta}_i(S'[B])$$

$$= \sum_{j=0}^{m-1} \left[\frac{1}{\binom{m}{j}} + \frac{1}{\binom{m}{j+1}} \right] \sum_{B \in \binom{V'}{j}} \widetilde{\beta}_i(S'[B]) = \frac{m+1}{m} \sigma_i(S').$$

Since $i \leq l \leq d - k - 1 < d$, $\widetilde{\beta}_i(S') = 0$. This justifies the first and third equalities above. The second equality holds since, by definition, $S'[A] = S'[B]$, where $B = A \cap V'$, and the map $A \mapsto A \cap V'$ is a bijection between $\binom{V}{j}$ and $\binom{V'}{j} \sqcup \binom{V'}{j-1}$. The last equality is because of the trivial identity $\frac{1}{\binom{m}{j}} + \frac{1}{\binom{m}{j+1}} = \frac{m+1}{m} \frac{1}{\binom{m-1}{j}}$. So, we have (when $t = 0$):

$$\sum_{i=0}^{l} (-1)^{l-i} \sigma_i(S) = (-1)^l \frac{m+1}{(d+3)(d+2)} + \frac{m+1}{m} \sum_{i=0}^{l} (-1)^{l-i} \sigma_i(S'), \ 0 \leq l \leq d - k - 1.$$

Now, since S' has $m - 1$ vertices, induction hypothesis gives

$$\sum_{i=0}^{l} (-1)^{l-i} \sigma_i(S') \leq \frac{m}{d+3} \sum_{i=0}^{l+1} (-1)^{l+1-i} \frac{g_i(S')}{\binom{d+2}{i}},$$

with equality for $k - 1 \leq l \leq d - k - 1$. Therefore, we get

$$\sum_{i=0}^{l} (-1)^{l-i} \sigma_i(S) \leq (-1)^l \frac{m+1}{(d+3)(d+2)} + \frac{m+1}{d+3} \sum_{i=0}^{l+1} (-1)^{l+1-i} \frac{g_i(S')}{\binom{d+2}{i}}$$

$$= \frac{m+1}{d+3} \sum_{i=0}^{l+1} (-1)^{l+1-i} \frac{g_i(S') + \delta_{i1}}{\binom{d+2}{i}}$$

$$= \frac{m+1}{d+3} \sum_{i=0}^{l+1} (-1)^{l+1-i} \frac{g_i(S)}{\binom{d+2}{i}},$$

with equality for $k - 1 \leq l \leq d - k - 1$. This completes the induction in the last case, thus proving (b) and (c). $\square$

Now, the following key result on the mu-vector of 2-neighbourly members of $\mathcal{W}_k(d)$ is more or less immediate.

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Proposition 5.4. *Let $M \in \mathcal{W}_k(d)$ be 2-neighbourly with $d \geq 2k \geq 2$. Then the mu-vector of M (with respect to any field) is related to its g-vector as follows:*

(a) $\mu_i = 0$ *for* $k+1 \leq i \leq d-k-1$,

(b) $\displaystyle\sum_{i=1}^{l}(-1)^{l-i}\mu_i \leq \frac{g_{l+1}}{\binom{d+2}{l+1}}$ *for* $1 \leq l \leq k-1$, *and*

(c) $\displaystyle\sum_{i=1}^{l}(-1)^{l-i}\mu_i = \frac{g_{l+1}}{\binom{d+2}{l+1}}$, *for* $k \leq l \leq d-k-1$.

Proof. Let m be the number of vertices of M and let L_x be the link of x in M for $x \in V(M)$. Then each L_x is a k-stellated sphere, of dimension $d-1 \geq 2k-1$, on exactly $m-1$ vertices. Therefore, for $k+1 \leq i \leq d-k-1$, $\sigma_{i-1}(L_x) = 0$ by Proposition 5.3 (a). Taking the sum of these equations over all $x \in V(M)$, we get $\mu_i(M) = 0$ for $k+1 \leq i \leq d-k-1$. This proves part (a).

Also, by Proposition 5.3 (b) and (c), we have, for $1 \leq l \leq d-k-1$,

$$\sum_{i=1}^{l}(-1)^{l-i}\sigma_{i-1}(L_x) \leq \frac{m}{d+2}\sum_{i=0}^{l}(-1)^{l-i}\frac{g_i(L_x)}{\binom{d+1}{i}},$$

with equality for $k \leq l \leq d-k-1$. Adding these over all $x \in V(M)$, and dividing the result by m, we get (in view of Lemma 4.7):

$$\sum_{i=1}^{l}(-1)^{l-i}(\mu_i - \delta_{i1}) \leq \frac{1}{d+2}\sum_{i=0}^{l}\frac{(-1)^{l-i}}{\binom{d+1}{i}}((d+2-i)g_i + (i+1)g_{i+1}),$$

with equality for $k \leq l \leq d-k-1$. That is,

$$\sum_{i=1}^{l}(-1)^{l-i}\mu_i \leq (-1)^{l-1} + \sum_{i=0}^{l}(-1)^{l-i}\left(\frac{g_i}{\binom{d+2}{i}} + \frac{g_{i+1}}{\binom{d+2}{i+1}}\right) = \frac{g_{l+1}}{\binom{d+2}{l+1}} \text{ (since } g_0 = 1),$$

with equality for $k \leq l \leq d-k-1$. This proves (b) and (c). $\qquad\square$

Now, we can prove one of the main results of this paper.

Proposition 5.5 (A lower bound theorem for $\mathcal{W}_k(d)$). *Let $M \in \mathcal{W}_k(d)$ be 2-neighbourly. Then the g-vector of M is related to its Betti numbers (with respect to any field) as follows:*

(a) *if* $d = 2k$, *then* $g_{l+1} \geq \binom{d+2}{l+1}\displaystyle\sum_{i=1}^{l}(-1)^{l-i}\beta_i$ *for* $1 \leq l \leq k-1$,

(b) *if* $d \geq 2k+1$, *then* $g_{l+1} \geq \binom{d+2}{l+1}\displaystyle\sum_{i=1}^{l}(-1)^{l-i}\beta_i$ *for* $1 \leq l \leq k$,

(c) *if* $d \geq 2k+2$, *then* $g_{l+1} = \binom{d+2}{l+1}\displaystyle\sum_{i=1}^{l}(-1)^{l-i}\beta_i$ *for* $k \leq l \leq d-k-1$, *and*

(d) *if* $d \geq 2k+2$, *then* $\beta_i = 0$ *for* $k+1 \leq i \leq d-k-1$,

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Proof. By Propositions 4.13 (b) and 5.4 (a), we have $0 \leq \beta_i \leq \mu_i = 0$ for $k+1 \leq i \leq d-k-1$. This proves part (d).

Since M is 2-neighbourly, it is connected. Therefore, $\beta_0 = 1 = \mu_0$. Hence, Proposition 4.13 (a) yields $\sum_{i=1}^{l}(-1)^{l-i}\beta_i \leq \sum_{i=1}^{l}(-1)^{l-i}\mu_i$ for $l \geq 1$. Therefore, parts (a) and (b) are immediate from Proposition 5.4 (b) and (c).

If $k < l \leq d - k - 1$, then $\beta_l = \mu_l$ by part (d), and hence by Proposition 4.13 (d), $H_l(Y) \to H_l(M)$ is injective for any induced subcomplex Y of M. Hence, by Proposition 4.13 (c), $\sum_{i=1}^{l}(-1)^{l-i}\beta_i = \sum_{i=1}^{l}(-1)^{l-i}\mu_i$. But, $\sum_{i=1}^{l}(-1)^{l-i}\mu_i = g_{l+1}/\binom{d+2}{l+1}$ by Proposition 5.4 (c). This proves part (c) in case $k < l \leq d-k-1$. Finally, when $k+1 \leq d-k-1$, we have $\beta_{k+1} = 0 = \mu_{k+1}$ by part (d) and Proposition 5.4 (a), hence $H_k(Y) \to H_k(M)$ is injective for any induced subcomplex Y of M (by Proposition 4.13 (d)). Therefore, by Propositions 4.13 (c) and 5.4 (c), $\sum_{i=1}^{k}(-1)^{k-i}\beta_i = \sum_{i=1}^{k}(-1)^{k-i}\mu_i = g_{k+1}/\binom{d+2}{k+1}$, completing the proof of part (c). $\square$

We also need the following elementary result.

Lemma 5.6. *Let X be an $(l + 1)$-neighbourly simplicial complex. Then the beta- and mu-vectors of X (with respect to any field) satisfy $\beta_i = 0 = \mu_i$ for $1 \leq i \leq l - 1$.*

Proof. The l-skeleton of X agrees with that of the standard ball of dimension $f_0(X) - 1$. Since the ball is homologically trivial and the ith homology of a simplicial complex is the same as the ith homology of its $(i + 1)$-skeleton, it follows that $\beta_i(X) = 0$ for $1 \leq i \leq l - 1$.

Also, for any vertex x of X, the link L_x of x in X is l-neighbourly. Therefore, by the same argument, we have, for $1 \leq i \leq l - 1$, $\widetilde{\beta}_{i-1}$ of any induced subcomplex of L_x is $= 0$, except that $\widetilde{\beta}_0 = -1$ for the empty subcomplex. Therefore, taking an appropriate weighted sum, we get $\sigma_{i-1}(L_x) = -\delta_{i1}$ for $1 \leq i \leq l - 1$ and $x \in V(X)$. Adding over all $x \in V(X)$, we get $\mu_i(X) = 0$ for $1 \leq i \leq l - 1$. $\square$

The following result (which is the first known combinatorial criterion for tightness) is due to Kühnel [25].

Lemma 5.7 (Kühnel). *For $k \geq 1$, let M be a $(k + 1)$-neighbourly triangulation of an $\mathbb{F}$-orientable closed manifold of dimension $2k$. Then M is $\mathbb{F}$-tight.*

(Note that, when $k \geq 2$, M is by assumption at least 3-neighbourly, and hence simply connected. Thus, the hypothesis of orientability is automatic for $k \geq 2$.)

Proof. Since M is at least 2-neighbourly, it is connected. Therefore, $\mu_0 = 1 = \beta_0$. By Lemma 5.6, $\mu_i = 0 = \beta_i$ for $1 \leq i \leq k-1$. So, by duality (Proposition 4.13 (e)), $\mu_i = 0 = \beta_i$ for $k + 1 \leq i \leq 2k - 1$ and $\mu_{2k} = 1 = \beta_{2k}$. Thus $\mu_i = \beta_i$ for all i, except possibly for $i = k$. But then, the equality $\sum_{i=0}^{2k}(-1)^i\mu_i = \sum_{i=0}^{2k}(-1)^i\beta_i$ from Proposition 4.13 (a) implies $\mu_i = \beta_i$ for $i = k$ as well. Therefore, by Proposition 4.16, M is $\mathbb{F}$-tight. $\square$

The "if" part of Proposition 5.8 (a) below is essentially due to Effenberger [17]. This paper was largely motivated by a desire to understand and generalize Effenberger's result. (In this connection, recall that $\mathcal{W}_1(d) = \mathcal{K}_1(d)$ by Corollary 2.16.)

Proposition 5.8. *Let $M \in \mathcal{W}_1(d)$. Then we have the following.*

(a) *If $d \neq 3$, then M is $\mathbb{F}$-tight if and only if M is 2-neighbourly and $\mathbb{F}$-orientable.*

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(b) *If $d = 3$, then M is $\mathbb{F}$ tight if and only if M is 2-neighbourly, $\mathbb{F}$-orientable, and satisfies $\beta_1(M; \mathbb{F}) = (n-4)(n-5)/20$, where $n = f_0(M)$.*

Proof. By Proposition 4.15, to be $\mathbb{F}$-tight, M must be 2-neighbourly and $\mathbb{F}$-orientable. Also, if $d = 3$ and M is 2-neighbourly on n vertices, then (from (4)) $g_2 = \binom{n-4}{2}$. By Proposition 5.4 (c), $\mu_1(M) = g_2(M)/10 = (n-4)(n-5)/20$. Therefore, for M to be $\mathbb{F}$-tight, Proposition 4.16 requires $\beta_1(M) = (n-4)(n-5)/20$. Thus, we have the "only if" part of (a) and (b).

Now, we prove the "if" parts. If $d = 1$, the result is trivial since S_3^1 is the only 2-neighbourly closed 1-manifold. If $d = 2$, the result is immediate from Lemma 5.7. If $d = 3$, we have $\mu_0 = 1 = \beta_0$ because of connectedness, and hence $\mu_3 = 1 = \beta_3$ by duality. Also, $\mu_1 = g_2/\binom{d+2}{2} = \beta_1$ by Proposition 5.4 (c) and hypothesis. Therefore, $\mu_2 = g_2/\binom{d+2}{2} = \beta_2$ by duality. Hence, by Proposition 4.16, M is $\mathbb{F}$-tight. So, assume that $d \geq 4$. Then $\mu_0 = 1 = \beta_0$ and hence $\mu_d = 1 = \beta_d$. By Propositions 5.4 (c) and 5.5 (c), $\mu_1 = g_2/\binom{d+2}{2} = \beta_1$, and hence $\mu_{d-1} = g_2/\binom{d+2}{2} = \beta_{d-1}$ by duality. Also, by Propositions 5.4 (a) and 5.5 (d), $\mu_i = 0 = \beta_i$ for $2 \leq i \leq d - 2$. Thus, $\mu_i = \beta_i$ for all i. Hence M is $\mathbb{F}$-tight by Proposition 4.16. $\square$

The "$d = 4$" case of the following result is due to Walkup [41] and Kühnel [25] (cf. [5, Proposition 2]). Part (b) of this proposition is Theorem 5 of Lutz, Sulanke and Swartz [31].

Proposition 5.9 (A lower bound theorem for triangulated manifolds). *Let M be a connected closed triangulated manifold of dimension $d \geq 3$. Let $\beta_1 = \beta_1(M; \mathbb{Z}_2)$. Then the face vector of M satisfies:*

(a) $f_j \geq \begin{cases} \binom{d+1}{j} f_0 + j \binom{d+2}{j+1}(\beta_1 - 1), & \text{if } 1 \leq j < d, \\ d f_0 + (d-1)(d+2)(\beta_1 - 1), & \text{if } j = d. \end{cases}$

(b) $\binom{f_0 - d - 1}{2} \geq \binom{d+2}{2}\beta_1$.

When $d \geq 4$, equality holds in (a) (for some $j \geq 1$, equivalently, for all j) if and only if $M \in \mathcal{W}_1(d)$, and equality holds in (b) if and only if M is a 2-neighbourly member of $\mathcal{W}_1(d)$.

Proof. Using Corollary 4.4 and formulae (5) and (6), it is easy to see that the face vector of any 1-stellated $(d-1)$-sphere S is given by

$$f_{j-1}(S) = \begin{cases} \binom{d}{j-1} f_0(S) - (j-1)\binom{d+1}{j}, & \text{if } 1 \leq j < d, \\ (d-1)f_0(S) - (d-2)(d+1), & \text{if } j = d. \end{cases}$$

The lower bound theorem of Kalai ([21]) says that if L is any triangulated closed manifold of dimension $d - 1$ with $f_0(L) = f_0(S)$, then $f_{j-1}(L) \geq f_{j-1}(S)$ for $1 \leq j \leq d$. Also, when $d \geq 4$, equality holds here (for some $j > 1$, equivalently, for all j) if and only if L is an 1-stellated sphere. Applying this result to the vertex links L_x, $x \in V(M)$, of M, we get

$$f_{j-1}(L_x) \geq \begin{cases} \binom{d}{j-1} f_0(L_x) - (j-1)\binom{d+1}{j}, & \text{if } 1 \leq j < d, \\ (d-1)f_0(L_x) - (d-2)(d+1), & \text{if } j = d. \end{cases}$$

But $\sum_{x \in V(M)} f_{j-1}(L_x) = (j+1)f_j$, $1 \leq j \leq d$. Therefore, adding the above inequalities over all vertices x of M, we get that the face vector of M satisfies:

$$f_j \geq \begin{cases} \frac{2}{j+1}\binom{d}{j-1}f_1 - \frac{j-1}{j+1}\binom{d+1}{j}f_0, & \text{if } 1 \leq j < d, \\ \frac{2d-2}{d+1}f_1 - (d-2)f_0, & \text{if } j = d. \end{cases} \tag{13}$$

Also, when $d \geq 4$, equality holds in (13) for some $j > 1$ (equivalently, for all j) if and only if all the vertex links of M are 1-stellated, i.e., if and only if $M \in \mathcal{W}_1(d)$.

Now, Theorem 5.2 of Novik and Swartz [35] says that $g_2(M) \geq \binom{d+2}{2}\beta_1$, i.e.,

$$f_1 \geq (d+1)f_0 + \binom{d+2}{2}(\beta_1 - 1). \tag{14}$$

Also, when $d \geq 4$, equality holds in (14) if and only if $M \in \mathcal{W}_1(d)$. Notice that (14) is just the case $j = 1$ of part (a). Now, combining (13) and (14), we get all cases of part (a), after a little simplification.

We also have $(d+1)f_0 + \binom{d+2}{2}(\beta_1 - 1) \leq f_1 \leq \frac{f_0(f_0-1)}{2}$, where the first inequality is from (14) (with equality for $d \geq 4$ if and only if $M \in \mathcal{W}_1(d)$) and the second inequality is trivial (with equality if and only if M is 2-neighbourly). Hence we have $(d+1)f_0 + \binom{d+2}{2}(\beta_1 - 1) \leq \frac{f_0(f_0-1)}{2}$, which simplifies to the inequality in part (b). $\square$

Next we introduce:

Definition 5.10. A d-dimensional simplicial complex X is said to be *minimal* if $f_0(X) \leq f_0(Y)$ for every triangulation Y of the geometric carrier $|X|$ of X. We shall say that X is *strongly minimal* if $f_i(X) \leq f_i(Y)$, $0 \leq i \leq d$, for all such Y.

In [27], Kühnel and Lutz conjectured that all $\mathbb{F}$-tight triangulated manifolds are strongly minimal. Our next result is a powerful evidence in favour of this conjecture. It also generalizes a result of Swartz [40, Theorem 4.7], who proved that the tight triangulations $K_{2d+3}^d \in \mathcal{W}_1(d)$ (cf. Example 6.11 (b) below) are strongly minimal for all $d \geq 2$.

Corollary 5.11. *Every $\mathbb{F}$-tight member of $\mathcal{W}_1(d)$ is strongly minimal.*

Proof. Let $M_0 \in \mathcal{W}_1(d)$ be $\mathbb{F}$-tight. Proposition 5.8 implies that M_0 is $\mathbb{Z}_2$-tight. By the same proposition, M_0 is 2-neighbourly. Now, for $d \leq 2$, any 2-neighbourly closed d-manifold is strongly minimal. (This is entirely trivial for $d = 1$ since S_3^1 is the only 2-neighbourly closed manifold in this case. It is only a little less trivial for $d = 2$: the face vector is determined by f_0 and β_1 in this case.) So, assume $d \geq 3$.

We claim that M_0 attains all the bounds in Proposition 5.9. This is immediate from the proposition itself when $d \geq 4$. If $d = 3$, then - as $M_0 \in \mathcal{W}_1(3)$ is $\mathbb{Z}_2$-tight, we have $g_2(M_0) = 10\beta_1$ by Proposition 5.8. Hence, following the proof of Proposition 5.9, one sees that M_0 attains the bounds in Proposition 5.9 in this case also.

Now, let M be any triangulation of $|M_0|$. Since the Betti numbers are topological invariants, we have $\beta_1(M; \mathbb{Z}_2) = \beta_1(M_0; \mathbb{Z}_2) = \beta_1$ (say). By Proposition 5.9 and the above claim,

$$\binom{f_0(M) - d - 1}{2} \geq \binom{d+2}{2}\beta_1 = \binom{f_0(M_0) - d - 1}{2}.$$

Since, trivially, $f_0(M)$, $f_0(M_0) \geq d + 2$, this implies $f_0(M) \geq f_0(M_0)$. Therefore, we get:
$f_j(M) \geq a_j f_0(M) + b_j \geq a_j f_0(M_0) + b_j = f_j(M_0)$ for $0 \leq j \leq d$, where $a_j > 0$ and b_j are constants (depending only on d, j and β_1) given by Proposition 5.9. $\qquad\square$

2-neighbourly members of $\mathcal{W}_1(d)$ were called "*tight neighbourly*" by Lutz, Sulanke and Swartz [31]. By Proposition 5.8, tight neighbourly manifolds of dimension $\neq 3$ are $\mathbb{Z}_2$-tight if non-orientable, and tight if orientable. Part (b) of Proposition 5.8 gives a criterion for the tightness of a tight neighbourly 3-manifold in terms of its first Betti number. Corollary 5.11 shows that tight neighbourly triangulations of dimension ≥ 4 are strongly minimal.

In consequence of Corollary 4.6, a member of $\mathcal{W}_k(d)$ can be at most $(k+1)$-neighbourly, unless it is a standard sphere. This observation leads us to introduce:

Notation: $\mathcal{W}_k^*(d)$ will denote the subclass of $\mathcal{W}_k(d)$ consisting of all the $(k+1)$-neighbourly members of the latter class.

Thus, Proposition 5.8 was about the class $\mathcal{W}_1^*(d)$, the class of tight neighbourly d-manifolds. By a result of Kalai (cf. [21, Corollary 8.4] or [5, Proposition 3]), for $d \geq 4$, any member of $\mathcal{W}_1(d)$ triangulates the connected sum of finitely many copies of $S^1 \times S^{d-1}$ or of $S^{d-1} \times S^1$. The following result may be compared with Kalai's result.

Proposition 5.12. *Let $M \in \mathcal{W}_k^*(d)$, where $k \geq 2$ and $d \geq 2k + 2$. Suppose M is not a $\mathbb{Z}$-homology sphere. Then M has the same integral homology group as the connected sum of β copies of $S^k \times S^{d-k}$, where the positive integer β is given in terms of the number m of vertices of M by the formula $\beta = \binom{m+k-d-2}{k+1}/\binom{d+2}{k+1}$. In consequence, we must have $m \geq 2d + 4 - k$ and $\binom{d+2}{k+1}$ divides $\binom{m+k-d-2}{k+1}$.*

(As to the inequality $m \geq 2d + 4 - k$, note that by a result of Brehm and Kühnel [8], this lower bound on the number of vertices holds, more generally, for any triangulation of a closed d-manifold which is not k-connected.)

Proof. Since M is at least 3-neighbourly, it is simply connected and hence orientable. Therefore, by Poincaré duality, the Betti numbers of M with respect to any field $\mathbb{F}$ satisfy $\beta_{d-i} = \beta_i$, $0 \leq i \leq d$. Since M is connected, we have $\beta_0 = 1$ and hence $\beta_d = 1$. By Lemma 5.6, $\beta_i = 0$ for $1 \leq i \leq k-1$. Hence, by duality, $\beta_i = 0$ for $d - k + 1 \leq i \leq d - 1$. By Proposition 5.5 and duality, $\beta_{d-k} = \beta_k = g_{k+1}/\binom{d+2}{k+1}$ and $\beta_i = 0$ for $k + 1 \leq i \leq d - k - 1$. Thus, all the Betti numbers of M are independent of the choice of the field $\mathbb{F}$. Therefore, by the universal coefficients theorem, the $\mathbb{Z}$-homologies of M are torsion free, and the $\mathbb{Z}$-Betti numbers are given by the same formulae as above. Since all the middle Betti numbers except possibly $\beta_k = \beta_{d-k}$ are zero, it follows that if M is not a homology sphere, then the value $\beta = g_{k+1}/\binom{d+2}{k+1} = \binom{m+k-d-2}{k+1}/\binom{d+2}{k+1}$ of β_k must be a strictly positive integer. The last statement follows from this observation. $\qquad\square$

Remark 5.13. If $M \in \mathcal{W}_k^*(d)$ is a homology sphere ($k \geq 2$, $d \geq 2k+2$), then - by the above proof - we must have $g_{k+1}(M) = 0$. In this case, we expect (GLBC) M to be a k-stacked d-sphere. Since S_{d+2}^d is the only $(k+1)$-neighbourly k-stacked d-sphere, we should therefore have $M = S_{d+2}^d$ in this case. Thus, it should be possible to replace the hypothesis "M is not an integral homology sphere" in Proposition 5.12 by the simpler hypothesis $M \neq S_{d+2}^d$.

The next result is our generalization of Proposition 5.8 to the case $k \geq 2$.

Proposition 5.14 (A combinatorial criterion for tightness). *Let $M \in \mathcal{W}_k^*(d)$, where $k \geq 2$. Then we have :*

(a) *if $d \neq 2k + 1$ then M is tight, and*

(b) *if $d = 2k + 1$, then M is $\mathbb{F}$-tight if and only if $\beta_k(M; \mathbb{F}) = \dfrac{\binom{n-k-3}{k+1}}{\binom{2k+3}{k+1}}$, where $n = f_0(M)$.*

Proof. This is trivial if $M = S_{d+2}^d$. So, assume $M \neq S_{d+2}^d$. Then there is a vertex x of M such that the link of x in M (is a k-neighbourly $(d-1)$-sphere which) is not a standard sphere. Therefore, by Remark 4.5 (c), $d - 1 \geq 2k - 1$. Thus $d \geq 2k$. If $d = 2k$, then the result follows from Lemma 5.7. So, assume that $d \geq 2k + 1$.

Notice that M is at least 3-neighbourly, hence orientable. Thus the duality result of Proposition 4.13 (e) applies. Since M is connected, we have $\beta_0 = 1 = \mu_0$ and hence $\beta_d = 1 = \mu_d$. By Lemma 5.6, $\beta_i = 0 = \mu_i$ for $1 \leq i \leq k-1$, hence also for $d-k+1 \leq i \leq d-1$. Since M is $(k+1)$-neighbourly, we have $g_{k+1}(X) = \binom{n+k-d-2}{k+1}$, where $n = f_0(M)$. We also have $\beta_k = g_{k+1}/\binom{d+2}{k+1} = \mu_k$ (by hypothesis and Proposition 5.4 (c) when $d = 2k + 1$; by Propositions 5.4 (c) and 5.5 (c) when $d \geq 2k + 2$). Hence, $\beta_{d-k} = g_{k+1}/\binom{d+2}{k+1} = \mu_{d-k}$. By Propositions 5.4 (a) and 5.5 (d), we also have $\beta_i = 0 = \mu_i$ for $k + 1 \leq i \leq d - k - 1$. Thus, $\beta_i = \mu_i$ for all i. Hence M is $\mathbb{F}$-tight by Proposition 4.16 (and, when $d \neq 2k + 1$, this argument applies to all fields $\mathbb{F}$).

For the converse statement in part (b), note that - more generally - for any $M \in \mathcal{W}_k^*(d)$ with $d \geq 2k+1$, Lemma 5.6 and Proposition 5.4 (c) imply that $\mu_k = g_{k+1}/\binom{d+2}{k+1}$. Therefore, for M to be $\mathbb{F}$-tight, we must have (by Proposition 4.16) $\beta_k = g_{k+1}/\binom{d+2}{k+1}$ as well. Thus, for $d = 2k + 1$, $\beta_k = \binom{n+k-d-2}{k+1}/\binom{d+2}{k+1} = \binom{n-k-3}{k+1}/\binom{2k+3}{k+1}$. $\qquad\square$

Propositions 5.4 and 5.5 may be used to prove many more tightness criteria. For instance, we have :

Proposition 5.15 (Another tightness criterion?). *Let $k \geq 2$, and let $M \in \mathcal{W}_k(d)$ be k-neighbourly of dimension $d = 2k$ or $d \geq 2k + 2$, on n vertices. If M is $\mathbb{F}$-orientable and $\beta_{k-1}(M; \mathbb{F}) = \binom{n+k-d-3}{k}/\binom{d+2}{k}$ then M is $\mathbb{F}$-tight.*

(Note that the requirement of $\mathbb{F}$-orientability is automatically fulfilled if $\mathbb{F} = \mathbb{Z}_2$ or $k \geq 3$.)

Proof. Since M is k-neighbourly and $k \geq 2$, we have $\mu_0 = 1 = \beta_0$ and $\mu_i = 0 = \beta_i$ for $1 \leq i \leq k-2$. Hence, by duality, $\mu_d = 1 = \beta_d$ and $\mu_i = 0 = \beta_i$ for $d-k+2 \leq i \leq d-1$. Also, as M is a k-neighbourly d-manifold on n vertices, it has $g_k = \binom{n+k-d-3}{k}$. Thus we have $g_k/\binom{d+2}{k} = \beta_{k-1} \leq \mu_{k-1} \leq g_k/\binom{d+2}{k}$, where the equality is by hypothesis and the inequalities are from Propositions 4.13 (b) and 5.4 (b). Thus, $\mu_{d-k+1} = \mu_{k-1} = \beta_{k-1} = \beta_{d-k+1}$. So, we have $\mu_i = \beta_i$ for $0 \leq i \leq k - 1$ and for $d - k + 1 \leq i \leq d$. So, in view of Proposition 4.16, to complete the proof it is sufficient to show that $\mu_i = \beta_i$ for $k \leq i \leq d - k$ as well.

If $d = 2k$, then we have $\mu_i = \beta_i$ for all $i \neq k$. Hence, by the equality statement in Proposition 4.13 (a), we have $\mu_i = \beta_i$ for $i = k$ also.

Now, suppose $d \geq 2k + 2$. Then, by Propositions 5.4 (c) and 5.5 (c), $\mu_k - \mu_{k-1} = g_{k+1}/\binom{d+2}{k+1} = \beta_k - \beta_{k-1}$. Hence $\mu_k = \mu_{k-1} + g_{k+1}/\binom{d+2}{k+1} = g_k/\binom{d+2}{k} + g_{k+1}/\binom{d+2}{k+1} = \beta_{k-1} + g_{k+1}/\binom{d+2}{k+1} = \beta_k$. Therefore, by duality, $\mu_{d-k} = \beta_{d-k}$. Also, by Propositions 5.4 (a) and 5.5 (d), $\mu_i = 0 = \beta_i$ for $k + 1 \leq i \leq d - k - 1$. $\qquad\square$

Remark 5.16. Note that $\mathcal{W}_{k-1}(d) \subseteq \mathcal{W}_k(d)$. If $M \in \mathcal{W}_{k-1}(d)$ is k-neighbourly and $d \geq 2k$, then M is $\mathbb{F}$-tight (whenever it is $\mathbb{F}$-orientable) by Propositions 5.8 and 5.14. Also, in this case, we automatically have (by Lemma 5.6 and Proposition 5.5 (c)) $\beta_{k-1} = \binom{n+k-d-3}{k}/\binom{d+2}{k}$. Thus, Proposition 5.15 has new content only for $M \in \mathcal{W}_k(d) \setminus \mathcal{W}_{k-1}(d)$.

6 Examples, counterexamples, questions and conjectures

Example 6.1 (Stellated versus stacked spheres).

(a) Let $S_{2d+2}^d = (S_2^0)^{*d+1}$, the join of $d+1$ copies of S_2^0. Being the boundary complex of the $(d+1)$-dimensional cross polytope, S_{2d+2}^d is a polytopal d-sphere. Therefore, by Proposition 3.8 (with $k = 0$), it is d-stellated. Also, by Proposition 2.6, it is d-stacked. Since S_{2d+2}^d is the clique complex of its edge graph (1-skeleton), it is not $(d-1)$-stacked. (If there was a $(d-1)$-stacked $(d+1)$-ball B such that $\partial B = S_{2d+2}^d$, then all the faces of B would be cliques of the edge graph of S_{2d+2}^d. But, all such cliques are in S_{2d+2}^d itself.) For the same reason, S_{2d+2}^d does not contain any induced standard sphere except S_2^0. Therefore, it does not admit any bistellar move of index ≥ 2. Therefore, S_{2d+2}^d is not $(d-1)$-stellated. (By the comment following Corollary 2.12, any k-stellated d-sphere, excepting S_{d+2}^d, admits a bistellar move of index $> d - k$.)

It is easy see that all the induced subcomplexes of S_{2d+2}^d are spheres and balls. Also, for $-1 \leq i \leq d$, S_{2d+2}^d has exactly $\binom{d+1}{i+1}$ induced i-spheres, each isomorphic to S_{2i+2}^i. Hence one computes:

$$\sigma_i(S_{2d+2}^d) = \begin{cases} -\frac{2d}{2d+1}, & i = 0 \\ \frac{\binom{d+1}{i+1}}{\binom{2d+2}{2i+2}}, & 1 \leq i \leq d. \end{cases}$$

Hence one finds (since all the vertex links of S_{2d+2}^d are isomorphic to S_{2d}^{d-1}) that the mu-vector of S_{2d+2}^d is given by

$$\mu_i(S_{2d+2}^d) = \frac{\binom{d}{i}}{\binom{2d}{2i}} \quad \text{for} \ \ 0 \leq i \leq d.$$

Surprisingly the mu-vector of S_{2d+2}^d satisfies the duality relation $\mu_{d-i} \equiv \mu_i$, even though it is not 2-neighbourly. Also, as a curiosity, we find $\sum_{i=0}^{d}(-1)^{d-i}\mu_i(S_{2d+2}^d) = \frac{2d+1}{2d+2}\chi(S^d)$. Thus, S_{2d+2}^d fails the strong Morse inequalities (Proposition 4.13 (a)).

(b) It is more difficult to find examples of $(d+1)$-stellated d-spheres (i.e., combinatorial d-spheres) which are not d-stellated. The following example is due to Dougherty, Faber and Murphy [15].

Let S_{16}^3 be the pure 3-dimensional simplicial complex with vertex set $\mathbb{Z}_{16} = \mathbb{Z}/16\mathbb{Z}$ and an automorphism $i \mapsto i + 1 \pmod{16}$. Modulo this automorphism, the basic facets of S_{16}^3 are:

$$\{0,1,4,6\}, \{0,1,4,9\}, \{0,1,6,14\}, \{0,1,8,9\}, \{0,1,8,10\}, \{0,1,10,14\}, \{0,2,9,13\}.$$

Of these, the fourth facet generates an orbit of length 8, while each of the other facets generates an orbit of length 16. Thus, S_{16}^3 has $1 \times 8 + 6 \times 16 = 104$ facets. The face

vector of S^3_{16} is $(16, 120, 208, 104)$. Since $120 = \binom{16}{2}$, S^3_{16} is 2-neighbourly and hence it does not allow any bistellar 1-move. Also, it is easy to verify that S^3_{16} has no edge of (minimum) degree 3, so that it does not allow any bistellar move of index 2 or 3 either. (So, S^3_{16} is an *unflippable* 3-sphere in the sense of [15]: it does not allow any bistellar move of positive index.) Thus, S^3_{16} is not 3-stellated. (Being a combinatorial 3-sphere, it is of course 4-stellated.) Following the proof of Proposition 2.6, fix a vertex x of S^3_{16}, and let $B^4_{16} = \{\{x\} \sqcup \alpha \, : \, x \notin \alpha \in S^3_{16}\}$. Then B^4_{16} is a 4-ball with $\partial B^4_{16} = S^3_{16}$. Since S^3_{16} is 2-neighbourly, B^4_{16} is a 2-stacked ball, and hence S^3_{16} is an example of a 2-stacked 3-sphere which is not even 3-stellated. If B^4_{16} was shellable, then (by Proposition 2.7) it would be 2-shelled and hence (by Corollary 2.4) S^3_{16} would be 2-stellated. Thus, B^4_{16} is an example of a non-shellable 2-stacked ball.

(c) It is even more difficult to find examples of triangulated d-spheres which are not $(d+1)$-stellated (i.e., not combinatorial d-spheres). Trivially, all triangulated spheres of dimension $d \leq 3$ are combinatorial spheres. In [16] and [19], Edwards and Freedman proved that a triangulated homology manifold of dimension $d \geq 3$ is a triangulated manifold if and only if all its vertex links are simply connected. In conjunction with Perelman's theorem (3-dimensional Poincaré conjecture) this shows that all triangulated 4-manifolds are combinatorial manifolds. The (non-) existence of triangulated 4-spheres which are not combinatorial spheres is equivalent to the still unresolved 4-dimensional smooth Poincaré conjecture. (According to [2], any such 4-sphere would require at least 13 vertices.) Thus, $d = 5$ is the smallest dimension in which we may reasonably expect triangulated spheres which are not combinatorial spheres. The following 16-vertex triangulation Σ^3_{16} of the Poincaré (integral) homology 3-sphere was found by Björner and Lutz [7]. The vertices of Σ^3_{16} are $1, \ldots, 9, 1', \ldots, 7'$. Its facets are: 1249, $1246'$, $1265'$, $1266'$, $1295'$, $1343'$, $1346'$, $1371'$, $1373'$, $131'6'$, $1493'$, $1564'$, $1565'$, $1582'$, $1584'$, $152'5'$, $164'6'$, $1781'$, $1782'$, $172'3'$, $181'4'$, $192'3'$, $192'5'$, $11'4'6'$, $2351'$, $2352'$, $2371'$, $2374'$, $232'4'$, $2494'$, $242'4'$, $242'6'$, $2582'$, $2583'$, $251'3'$, $261'3'$, $261'5'$, $263'6'$, $2794'$, $2795'$, $271'5'$, $282'6'$, $283'6'$, $3455'$, $3456'$, $343'5'$, $351'6'$, $352'5'$, $373'4'$, $32'4'5'$, $33'4'5'$, 4567, $4565'$, $4576'$, $4672'$, $461'2'$, $461'5'$, $472'6'$, $4893'$, $4894'$, $481'4'$, $481'5'$, $483'5'$, $41'2'4'$, $5674'$, $5794'$, $5796'$, $5893'$, $5894'$, $591'3'$, $591'6'$, $672'3'$, $673'4'$, $61'2'3'$, $63'4'6'$, $781'5'$, $782'6'$, $785'6'$, $795'6'$, $83'5'6'$, $91'2'3'$, $91'2'7'$, $91'6'7'$, $92'5'7'$, $95'6'7'$, $1'2'4'7'$, $1'4'6'7'$, $2'4'5'7'$, $3'4'5'6'$, $4'5'6'7'$. The face vector of Σ^3_{16} is $(16, 106, 180, 90)$. Björner and Lutz conjectured that it is strongly minimal.

Note that the vertex $6'$ is adjacent with all other vertices in Σ^3_{16}. Let D^4_{16} be the 4-dimensional simplicial complex whose facets are $\alpha \cup \{6'\}$, where α ranges over all facets of Σ^3_{16} not containing the vertex $6'$. Define $S^5_{18} = \partial(D^4_{16} * B^1_2)$, the boundary of the join of D^4_{16} and an edge. Observe that, $|S^5_{18}|$ is the double suspension of the Poincaré homology sphere $|\Sigma^3_{16}|$. Therefore, by Cannon's double suspension theorem (cf. [10], actually Cannon's theorem is a straightforward consequence of the result of Edwards and Freedman quoted above), S^5_{18} is a triangulated 5-sphere. Since it has Σ^3_{16} as the link of an edge, S^5_{18} is not a combinatorial sphere.

Let $D^6_{18} = D^4_{16} * B^1_2$, $D^7_{19} = D^4_{16} * B^2_3$ and $S^6_{19} = \partial D^7_{19}$. By the above logic, S^6_{19} is a triangulated 6-sphere. Since D^6_{18} is the antistar of a vertex in S^6_{19}, it follows from Lemma 4.1 in [4] that D^6_{18} is a triangulated 6-ball. Since the vertex $6'$ is adjacent to all the vertices in Σ^3_{16}, the construction of D^6_{18} shows that all the 3-faces of D^6_{18} lie in its boundary. Thus, D^6_{18} is a 2-stacked triangulated ball. As $S^5_{18} = \partial D^6_{18}$, it follows that S^5_{18} is an example of 2-stacked 5-sphere which is not even 6-stellated.

(d) Let S be a triangulated d-sphere and B is a k-stacked ball such that $\partial B = S$. Then, for any $e \geq 0$, $B * B_{e+1}^e$ is a k-stacked ball, and hence $\partial(B * B_{e+1}^e)$ is a k-stacked $(d+e+1)$-sphere. Also, S is a combinatorial sphere if and only if $\partial(B * B_{e+1}^e)$ is so. Applying this construction to the pair (S_{18}^5, D_{18}^6) in example (c) above, we find that for each $d \geq 5$, there are 2-stacked triangulated d-spheres which are not even $(d+1)$-stellated.

Claim. If $B = B_{16}^4$ is as in example (b) above then $\partial(B * B_{e+1}^e)$ is unflippable.

For $e \geq 0$, let $\widetilde{B}_{e+17}^{e+5} := B_{16}^4 * B_{e+1}^e$ and $\widetilde{S}_{e+17}^{e+4} := \partial\widetilde{B}_{e+17}^{e+5}$. Thus, $\widetilde{S}_{e+17}^{e+4} = (S_{16}^3 * B_{e+1}^e) \cup (B_{16}^4 * S_{e+1}^{e-1})$. Since S_{16}^3 is 2-neighbourly, so is $\widetilde{S}_{e+17}^{e+4}$. Therefore, $\widetilde{S}_{e+17}^{e+4}$ does not admit any bistellar 1-move. Suppose, if possible, that $\alpha \mapsto \beta$ is a bistellar move of index ≥ 2 on $\widetilde{S}_{e+17}^{e+4}$. Thus, $\mathrm{lk}_{\widetilde{S}_{e+17}^{e+4}}(\alpha) = \partial\beta$ and $\dim(\beta) \geq 2$, $\beta \notin \widetilde{S}_{e+17}^{e+4}$. Write $\alpha = \alpha_1 \sqcup \alpha_2$, where α_1 is a face of B_{16}^4 and α_2 is a face of B_{e+1}^e. If α_1 is an interior face of B_{16}^4, then $\alpha_2 \in S_{e+1}^{e-1}$ and $\partial\beta = \mathrm{lk}_{\widetilde{S}_{e+17}^{e+4}}(\alpha) = \mathrm{lk}_{B_{16}^4}(\alpha_1) * \mathrm{lk}_{S_{e+1}^{e-1}}(\alpha_2)$. Since the standard sphere $\partial\beta$ can't be written as the join of two spheres, it follows that either α_1 is a facet of B_{16}^4 or α_2 is a facet of S_{e+1}^{e-1}. If α_1 is a facet of B_{16}^4, then $\partial\beta = \mathrm{lk}_{S_{e+1}^{e-1}}(\alpha_2)$ and hence $\beta \in B_{e+1}^e \subseteq \widetilde{S}_{e+17}^{e+4}$. This is a contradiction since $\overline{\alpha} * \partial\beta$ is an induced subcomplex of $\widetilde{S}_{e+17}^{e+4}$. So, α_2 is a facet of S_{e+1}^{e-1} and hence $\partial\beta = \mathrm{lk}_{B_{16}^4}(\alpha_1)$. Then, $2 \leq \dim(\alpha_1) = 4 - \dim(\beta) \leq 2$ and hence $\dim(\alpha_1) = \dim(\beta) = 2$. Let $\alpha_1 = xuv$ (where x is the fixed vertex chosen in S_{16}^3 to construct B_{16}^4). Then $\mathrm{lk}_{S_{16}^3}(uv) = \mathrm{lk}_{B_{16}^4}(\alpha_1) = \partial\beta$. This is not possible since S_{16}^3 does not contain any edge of degree 3.

Thus α_1 is a boundary face of B_{16}^4, i.e., $\alpha_1 \in S_{16}^3$. If α_2 is the facet of B_{e+1}^e then $\mathrm{lk}_{S_{16}^3}(\alpha_1) = \mathrm{lk}_{\widetilde{S}_{e+17}^{e+4}}(\alpha) = \partial\beta$. Hence $\dim(\alpha_1) \geq 2$ and therefore $\dim(\beta) \leq 1$, a contradiction. So, α_2 is not the facet of B_{e+1}^e (and hence $\mathrm{lk}_{B_{e+1}^e}(\alpha_2)$ is a standard ball). Thus, the ball $B_1 := \mathrm{lk}_{B_{16}^4}(\alpha_1) * \mathrm{lk}_{B_{e+1}^e}(\alpha_2)$ is a non-trivial join of balls, so that all the vertices of B_1 are in its boundary. But, $\partial B_1 = \mathrm{lk}_{\widetilde{S}_{e+17}^{e+4}}(\alpha) = \partial\beta$. Therefore, B_1 is the standard ball $\overline{\beta}$ and hence $\mathrm{lk}_{B_{16}^4}(\alpha_1)$ is a standard ball. Therefore, $\mathrm{lk}_{S_{16}^3}(\alpha_1)$ is a standard sphere and hence $\dim(\alpha_1) \geq 2$. So, $\mathrm{lk}_{B_{16}^4}(\alpha_1)$ is a standard ball of dimension ≤ 1, i.e., it is a vertex or an edge. Then the vertex set of $\mathrm{lk}_{B_{16}^4}(\alpha_1)$ is a face in S_{16}^3. So, the vertex set β of $\mathrm{lk}_{\widetilde{S}_{e+17}^{e+4}}(\alpha)$ is a face of $S_{16}^3 * B_{e+1}^e \subseteq \widetilde{S}_{e+17}^{e+4}$. Therefore, $\overline{\alpha} * \partial\beta$ is not an induced subcomplex of $\widetilde{S}_{e+17}^{e+4}$, a contradiction. Thus, for each $e \geq 0$, $\widetilde{S}_{e+17}^{e+4}$ is an unflippable combinatorial $(e+4)$-sphere.

From this claim, it follows that $\partial(B_{16}^4 * B_{d-3}^{d-4})$ is a combinatorial d-sphere which is not d-stellated. Since B_{16}^4 is a 2-stacked ball, it follows that $B_{16}^4 * B_{d-3}^{d-4}$ is also 2-stacked. This implies that $\partial(B_{16}^4 * B_{d-3}^{d-4})$ is a 2-stacked combinatorial d-sphere which is not d-stellated, for $d \geq 4$. From this and the observation in (b), we find that for each $d \geq 3$, there are 2-stacked combinatorial d-spheres which are not d-stellated.

Since the classes $\Sigma_k(d)$, $\mathcal{S}_k(d)$ are increasing in k, we get:

- *For $2 \leq k \leq l \leq d \geq 3$ there are k-stacked combinatorial d-spheres which are not l-stellated.*
- *For $2 \leq k \leq l \leq d+1 \geq 6$ there are k-stacked triangulated d-spheres which are not l-stellated.*

(e) Let S_{10}^3 be the pure simplicial complex of dimension three whose vertices are the digits $0, 1, \ldots, 9$ and whose facets are:

   0123, 1234, 2345, 3456, 4567, 5678, 6789, 0128, 0139, 0189, 0238, 0356, 0358, 0369,
   0568, 0689, 1248, 1349, 1457, 1458, 1467, 1469, 1578, 1679, 1789, 2358, 2458, 3469.

Let S_{10}^2 be the pure 2-dimensional subcomplex of S_{10}^3 whose facets are:

   012, 013, 023, 124, 134, 235, 245, 346, 356, 457, 467, 568, 578, 679, 689, 789.

Then S_{10}^3 is a triangulated 3-sphere, and S_{10}^2 is a triangulated 2-sphere embedded in S_{10}^3. Being two-sided in S_{10}^3, the "equatorial" S_{10}^2 divides S_{10}^3 into two closed "hemispheres", say B_1 and B_2. Of course, B_1 and B_2 are triangulated 3-balls. The facets of B_1 are the first seven facets of S_{10}^3, while the facets of B_2 are the remaining twentyone facets of S_{10}^3.

The dual graph of the 3-ball B_1 is visibly a path. So, by Proposition 2.13, B_1 is 1-stacked. Since (from the above discussion, or by direct verification) $\partial B_1 = S_{10}^2 = \partial B_2$, it follows that S_{10}^2 is 1-stellated. But, it also bounds the ball B_2 which is Ziegler's example [43] of a non-shellable 3-ball! (If α is a facet of a triangulated d-ball B, then one says α is an *ear* of B if $B \setminus \{\alpha\}$ is also a triangulated d-ball. Clearly, if B is shellable, then the last facet, added while obtaining B from B_{d+1}^d by a sequence of shelling moves, must be an ear of B. Thus, if B has no ears, then it must be non-shellable. Such balls are "strongly non-shellable" in the terminology of Ziegler. A facet α of B is an ear of B if and only if the induced subcomplex of ∂B on the vertex set α is a $(d-1)$-ball. Using this criterion, it is possible to verify that B_2 has no ears: it is strongly non-shellable.)

(f) The following example of a shellable 3-ball with a unique ear is due to Frank Lutz (personal communication). Consider the pure 3-dimensional 2-neighbourly simplicial complex S_8^3 with vertices $1, 2, \ldots, 8$ and facets

   1234, 2345, 3456, 4567, 5678, 1237, 1248, 1278, 1348, 1356,
   1357, 1368, 1568, 1578, 2357, 2457, 2467, 2468, 2678, 3468.

Let S_8^2 be the pure 2-dimensional subcomplex of S_8^3 with facets

   123, 124, 134, 235, 245, 346, 356, 457, 467, 568, 578, 678.

Again, S_8^2 is a triangulated 2-sphere embedded in the triangulated 3-sphere S_8^3. As in (e) above, S_8^2 divides S_8^3 into two 3-balls B_1 and B_2. The facets of B_1 are the first five facets of S_8^3, while the facets of B_2 are the remaining fifteen facets of S_8^3. Again, B_1 is an 1-stacked 3-ball since its dual graph is a path. We have $\partial B_1 = S_8^2 = \partial B_2$. Thus, S_8^2 is an 1-stellated sphere. The other ball B_2 bounded by S_8^2 is shellable (indeed, 2-shelled). (A shelling of B_2: 1357, 1356, 1368, 1348, 1248, 3468, 1568, 1578, 1278, 2468, 2678, 1237, 2467, 2357, 2457.) But, B_2 has only one ear, namely 2457.

Clearly, the class $\mathcal{S}_k(d)$ of k-stacked d-spheres is closed under connected sum. In consequence, the class $\Sigma_1(d)$ of 1-stellated d-spheres is closed under connected sums. However, consider the following construction. Take a standard 2-ball B_3^2 with a vertex set $\{a, b, c\}$ disjoint from $V(S_8^3)$, and form the join $B := B_2 * B_3^2$. Then B is a 2-shelled

6-ball with a unique ear $2457abc$. Thus, $S := \partial B$ is a 2-stellated 5-sphere. The facets $245abc$, $457abc$ are two of the facets of S in the unique ear of B. Take a vertex disjoint copy B' of B, and let $S' = \partial B'$, the corresponding copy of S. Let $1', \ldots, 8', a', b', c'$ be the vertices of B' corresponding to the vertices $1, \ldots, 8, a, b, c$ respectively. Form the connected sum $\widetilde{B} = B \# B'$ by doing the identifications $2 \equiv 2', 4 \equiv 4', 5 \equiv 5', a \equiv a', b \equiv b', c \equiv c'$. Then $\widetilde{B}$ is a 16-vertex non-shellable 2-stacked 6-ball. Let $\widetilde{S} = \partial \widetilde{B}$. Then $\widetilde{S}$ is a 16-vertex 2-stacked 5-sphere which is not 2-stellated (by Propositions 2.11 and 2.9). (It can be shown that $\widetilde{S}$ is 5-stellated.) But, $\widetilde{S} = S \# S'$, the connected sum of two 2-stellated 5-spheres. For $d \geq 5$, if we take B_{d-2}^{d-3} in place of B_3^2 in the above construction then, by the same argument, we get a d-sphere which is not 2-stellated and is the connected sum of two 2-stellated d-spheres. Thus

- For $d \geq 5$, the class $\Sigma_2(d)$ is not closed under connected sum.

By Proposition 2.9, all the k-stellated spheres of dimension $d \geq 2k - 1$ are k-stacked. But, we are so far unable to answer:

Question 6.2. Is there a k-stellated d-sphere which is not k-stacked?

Note that, by Propositions 2.6 and 2.9, for an affirmative answer to Question 6.2, we must have $k + 1 \leq d \leq 2k - 2$, and hence $k \geq 3$, $d \geq 4$.

Notice that any $(k+1)$-neighbourly d-sphere is (trivially) $(d-k)$-stacked. A comparison of this observation with Proposition 3.8 as well as a comparison between Proposition 2.11 and Corollary 3.6 leads us to a strong suspicion:

Conjecture 6.3. *For $d \geq 2k$, a polytopal d-sphere is k-stellated if (and only if) it is k-stacked. Equivalently (in view of Corollary 3.6 and Propositions 2.7 and 2.9), if S is a k-stacked polytopal sphere of dimension $d \geq 2k$, then the $(d+1)$-ball $\overline{S}$ (given by formula (1)) is shellable.*

Let $S = S_{k+1}^{k-1} * S_{k+1}^{k-1}$, $B_1 = S_{k+1}^{k-1} * B_{k+1}^k$ and $B_2 = B_{k+1}^k * S_{k+1}^{k-1}$. Then B_1, B_2 are k-stacked polytopal $2k$-balls with $\partial B_1 = S = \partial B_2$. Thus, S is a $(2k-1)$-dimensional k-neighbourly polytopal k-stacked sphere. Hence S is k-stellated by Proposition 3.8. Thus, S is an example of a $(2k-1)$-dimensional k-stellated polytopal sphere which bounds two distinct (though isomorphic) k-stacked balls. So, the bound $d \geq 2k$ in Proposition 2.11 and Corollary 3.6 is sharp.

A comparison of Propositions 2.10 and 2.11 above leads us to the following query.

Question 6.4. (a) Are there k-stacked balls B_1, B_2 of dimension $2k+1$ such that $B_1 \neq B_2$ but $\partial B_1 = \partial B_2$?

(b) If S is a k-stacked sphere of dimension $\geq 2k + 1$ then by Proposition 2.10 there is a unique k-stacked ball $\overline{S}$ such that $\partial \overline{S} = S$. Is $\overline{S}$ always given by the formula (1)?

Example 6.5 (The Klee-Novik construction). For $d \geq 1$, let S_{2d+4}^{d+1} be the join of $d + 2$ copies of S_2^0 with disjoint vertex sets $\{x_i, y_i\}$, $1 \leq i \leq d + 2$. Then S_{2d+4}^{d+1} is a triangulated sphere with missing edges $x_i y_i$, $1 \leq i \leq d + 2$ (cf. Example 6.1 (a)). Each of the 2^{d+2} facets of S_{2d+4}^{d+1} may be encoded by a sequence of $d + 2$ signs as follows. If σ is a facet, then for each index i $(1 \leq i \leq d + 2)$ σ contains either x_i or y_i, but not both. Put $\varepsilon_i = +$ if $x_i \in \sigma$ and $\varepsilon_i = -$ if $y_i \in \sigma$. Thus the sign sequence $(\varepsilon_1, \ldots, \varepsilon_{d+2})$ encodes

the facet σ. For $0 \le k \le d$, let $\overline{M}(k, d)$ be the pure $(d + 1)$-dimensional subcomplex of S_{2d+4}^{d+1} whose facets are those facets σ (of the latter complex) whose sign sequences have at most k sign changes. (A sign change in the sign sequence $(\varepsilon_1, \ldots, \varepsilon_{d+2})$ is an index $1 \le i \le d + 1$ such that $\varepsilon_{i+1} \ne \varepsilon_i$.) Then $\overline{M}(k, d)$ is a pseudomanifold with boundary. Klee and Novik [22] proved that $M(k, d) := \partial \overline{M}(k, d)$ is a triangulation of $S^k \times S^{d-k}$ for $0 \le k \le d$. (In their paper, Klee and Novik use the notation $B(k, d + 2)$ for $\overline{M}(k, d)$.) The authors of [22] observed that the permutation D, E and R are automorphisms of $\overline{M}(k, d)$ (and hence of $M(k, d)$), where $D = \prod_{j=1}^{d+2} (x_j, y_j)$, $E = \prod_{1 \le j < (d+3)/2} (x_j, x_{d+3-j})(y_j, y_{d+3-j})$ and $R = (x_1, \ldots, x_{d+2})(y_1, \ldots, y_{d+2})$ when k is even, $R = (x_1, \ldots, x_{d+2}, y_1, \ldots, y_{d+2})$ when k is odd. Clearly, these three automorphisms generate a vertex-transitive automorphism group of $\overline{M}(k, d)$. Therefore, the links in $\overline{M}(k, d)$ (or in $M(k, d)$) of all the vertices are isomorphic. The involution $A = \prod_{j \text{ even}} (x_j, y_j)$ is an isomorphism between $M(k, d)$ and $M(d - k, d)$. Therefore, in discussing these constructions we may (and do) assume $d \ge 2k$. (However, A is not an isomorphism between $\overline{M}(k, d)$ and $\overline{M}(d - k, d)$. Indeed, A maps $\overline{M}(k, d)$ to the "complement" of $\overline{M}(d - k, d)$ in S_{2d+4}^{d+1}.)

Let $I = \{1, 2, \ldots, d + 1\}$. Define the linear order $\prec$ on $\binom{I}{\le k}$ by: $A \prec B$ if either $\#(A) < \#(B)$ or else $\#(A) = \#(B)$, $A <_{\text{lex}} B$, where $<_{\text{lex}}$ is the usual lexicographic order. Let L be the link of the vertex x_{d+2} in $\overline{M}(k, d)$. Clearly, for each $A \in \binom{I}{\le k}$, there is a unique facet τ_A of L such that A is precisely the set of sign-changes corresponding to the facet $\tau_A \cup \{x_{d+2}\}$ of $\overline{M}(k, d)$. We may transfer the linear order $\prec$ to the set of facets of L via the bijection $A \mapsto \tau_A$. Then, Klee and Novik show in [22] that $\prec$ is a shelling order for L. Thus, L is a shellable d-ball. What is more, if $\#(A) = j \le k$ then the facet τ_A of L is obtained (from the d-ball with facets τ_B, $B \prec A$) by a shelling move of index $j - 1$. In consequence, L is a k-shelled d-ball. Since the automorphism group of $\overline{M}(k, d)$ is vertex transitive, it follows that all vertex links of $\overline{M}(k, d)$ are k-shelled d-balls. Thus, $\overline{M}(k, d)$ is a $(d + 1)$-manifold with boundary. Also, since the boundary of a k-shelled ball is a k-stellated sphere (Corollary 2.4), it follows that $M(k, d) = \partial \overline{M}(k, d)$ has k-stellated vertex links. Thus,

- $M(k, d) \in \mathcal{W}_k(d)$ for $d \ge 2k$.

Also note that, when $d \ge 2k + 1$, the vertex links of $\overline{M}(k, d)$ are the unique (Proposition 2.11) k-stacked balls bounded by the corresponding vertex links of $M(k, d)$. Therefore, $\overline{M}(k, d)$ is the unique $(d + 1)$-manifold $\overline{M}$ such that $\partial \overline{M} = M(k, d)$ and $\text{skel}_{d-k}(\overline{M}) = \text{skel}_{d-k}(M(k, d))$. (In consequence, when $d \ge 2k + 2$, $\overline{M}(k, d)$ may be recovered from $M(k, d)$ via the formula (2) above; cf. Proposition 2.19.) Therefore, for $d \ge 2k + 1$, every automorphism of $M(k, d)$ extends to an automorphism of $\overline{M}(k, d)$: they have the same automorphism group. However, it is elementary to verify that the full automorphism group of $\overline{M}(k, d)$ is of order $4d + 8$. (Since this group is transitive on the $2d + 4$ vertices, it suffices to show that the full stabilizer of the vertex x_{d+2} is of order 2. This is easy.) Thus,

- When $d \ge 2k + 1$, the full automorphism group of $M(k, d)$ is of order $4d + 8$ (namely, the group generated by D, E, R above).

This leaves open the following tantalizing question.

Question 6.6. What is the full automorphism group of $M(k, 2k)$?

Notice that the involution A defined above is also an automorphism of $M(k, 2k)$. However, since A maps $\overline{M}(k, 2k)$ to its complement in S_{2d+4}^{d+1}, A is not an automorphism of $\overline{M}(k, 2k)$. Therefore, $A \notin H := \langle D, E, R \rangle$. The automorphism A normalizes H, so that the group $G := \langle D, E, R, A \rangle$ is of order $2 \times \#(H) = 16(k + 1)$. We suspect that G is the full automorphism group of $M(k, 2k)$. Is it?

For any vertex x of $M(k, d)$, let L_x (respectively $\overline{L}_x$) be the link of x in $M(k, d)$ (respectively in $\overline{M}(k, d)$). By the discussion above, $\overline{L}_x$ is obtained from B_{d+1}^d by $\sum_{j=1}^k \binom{d+1}{j}$ shelling moves, of which $\binom{d+1}{j}$ are of index $j - 1$. Therefore, by Lemma 2.3, $L_x = \partial \overline{L}_x$ is obtained from S_{d+1}^{d-1} by $\sum_{j=1}^k \binom{d+1}{j}$ bistellar moves, of which $\binom{d+1}{j}$ are of index $j - 1$. Since L_x is a k-stellated sphere of dimension $d - 1 \geq 2k - 1$, Proposition 4.3 implies that $g_j(L_x) = \binom{d+1}{j}$ for $0 \leq j \leq k$. Since $M(k, d)$ has $2d + 4$ vertices, Lemma 4.7 implies that the g-vector of $M(k, d)$ satisfies the recurrence $(d + 2 - j)g_j + (j + 1)g_{j+1} = \binom{d+1}{j}(2d + 4)$, $0 \leq j \leq k$. Solving this recurrence relation (with initial condition $g_0 = 1$) we see that the g-vector of $M(k, d)$ satisfies $g_j = \binom{d+2}{j}$ for $0 \leq j \leq k+1$. Since $d \geq 2k$ and $M(k, d) \in \mathcal{W}_k(d)$, Corollary 4.9 determines the entire g-vector of $M(k, d)$ from the above computation. By Proposition 4.8, we find:

- $g_{l+1}(M(k, d)) = \begin{cases} \binom{d+2}{l+1} & \text{if } 0 \leq l \leq k, \\ (-1)^{l-k}\binom{d+2}{l+1} & \text{if } k + 1 \leq l \leq d - k. \end{cases}$

(The rest of the g-numbers may now be determined using Klee's formula, namely (7). These formulae are in agreement with Theorem 5.2 in [22], of course.)

The calculation above shows that, for $d \geq 2k$, $M(k, d)$ satisfies the inequalities in Proposition 5.5 (indeed, with equality for $k \leq l \leq d - k - 1$) even though $M(k, d)$ is not 2-neighbourly. (Actually, the k-skeleton of $M(k, d)$ agrees with that of S_{2d+4}^{d+1}). We suspect that the assumption of 2-neighbourliness in Proposition 5.5 should be removable. These and other examples lead us to posit:

Conjecture 6.7 (GLBC for triangulated manifolds). *Let M be any triangulation of a connected closed d-manifold. Then, for $1 \leq l \leq \frac{d-1}{2}$, the g-numbers of M satisfy $g_{l+1}(M) \geq \sum_{i=1}^l (-1)^{l-i} \beta_i(M; \mathbb{F})$. Further, equality holds here for some $l < \frac{d-1}{2}$ if and only if $M \in \mathcal{K}_l(d)$.*

Notice that Proposition 5.5 proves the inequality of Conjecture 6.7 and the "if" part of the equality case of the conjecture under the extra assumption that M is 2-neighbourly and all the vertex links of M are $\lfloor \frac{d-1}{2} \rfloor$-stellated. The Klee-Novik manifolds $M(k, d)$, $d \geq 2k$, satisfy all parts of Conjecture 6.7. The "$l = 1$" case of Conjecture 6.7 (with $\mathbb{F} = \mathbb{Q}$) was a conjecture of Kalai [21]; Novik and Swartz proved it (for any field $\mathbb{F}$, with the extra hypothesis that M is $\mathbb{F}$-orientable) in [35, Theorem 5.2]. Since Conjecture 6.7 includes the GLBC for homology spheres, we do not expect it to be settled in a hurry.

We expect that part (b) of Proposition 5.9 ($\equiv$ Theorem 5 in [31]) should generalize as follows (compare Kühnel's conjecture [30, Conjecture 18]):

Conjecture 6.8. *If M is an m-vertex connected closed triangulated d-manifold with Betti numbers β_i (with respect to some field) then $\binom{m+l-d-2}{l+1} \geq \binom{d+2}{l+1}\sum_{i=1}^l (-1)^{l-i}\beta_i$ for $1 \leq l \leq (d-1)/2$. Also, equality holds here for some $l < (d-1)/2$ if and only if $M \in \mathcal{K}_l^*(d)$.*

In this connection, we may ask:

Question 6.9. Is it true that the g-vector of any triangulated closed d-manifold on m vertices satisfies $g_{l+1} \leq \binom{m+l-d-2}{l+1}$, with equality (if and) only if the triangulation is $(l+1)$-neighbourly? If this is true, then, of course, Conjecture 6.7 implies Conjecture 6.8.

By Proposition 5.12, for $d \geq 2k+2 \geq 6$, any member of $\mathcal{W}_k^*(d)$ has the same $\mathbb{Z}$-homology as the connected sum of copies of $S^k \times S^{d-k}$. This raises the question:

Question 6.10. Is it true that for $d \geq 2k + 2$, any member of $\mathcal{W}_k^*(d)$ triangulates a connected sum of (the total spaces of) S^{d-k}-bundles over S^k?

Example 6.11 (Tight triangulations of closed manifolds). We have noted that S^d_{d+2} is the only tight triangulation of S^d. This trivial series apart, we know the following examples of tight triangulations (cf. [27]).

(a) By Lemma 5.7, all 2-neighbourly 2-dimensional closed triangulated manifolds are tight when orientable and $\mathbb{Z}_2$-tight when non-orientable. For $n \geq 4$, there exist n-vertex 2-neighbourly orientable (respectively, non-orientable) triangulated 2-manifolds if and only if $n \equiv 0, 3, 4$ or 7 (mod 12) (respectively, $n \equiv 0$ or 1 (mod 3), except for $n = 4, 7$) (cf. [37]).

(b) For each $d \geq 2$, there is a $(2d + 3)$-vertex member K^d_{2d+3} of $\mathcal{W}_1^*(d)$ found by Kühnel [24]. For $d \geq 3$, it is the unique non-simply connected d-manifold on $2d + 3$ vertices (cf. [3, 12]). It is orientable (triangulates $S^{d-1} \times S^1$) for d even and non-orientable (triangulates $S^{d-1} \ltimes S^1$) for d odd. By Proposition 5.8, K^d_{2d+3} is tight for d even, and $\mathbb{Z}_2$-tight for d odd.

(c)   (i) The 15-vertex triangulation of $(S^3 \ltimes S^1)^{\#3}$ obtained (in [5]) is in $\mathcal{W}_1^*(4)$, hence $\mathbb{Z}_2$-tight by Proposition 5.8.

  (ii) Recently, Nitin Singh, a student of the second author, modified this construction to obtain (in [38]) two 15-vertex triangulations of $(S^3 \times S^1)^{\#3}$ in $\mathcal{W}_1^*(4)$. Both are tight by Proposition 5.8.

(d) Lutz constructed (in [29]) two 12-vertex triangulations of $S^2 \times S^3$; they belongs to $\mathcal{W}_2^*(5)$. By Proposition 5.14, these are tight triangulations.

(e) Only finitely many $2k$-dimensional $(k + 1)$-neighbourly triangulated closed manifolds are known for $k \geq 2$. By Lemma 5.7, they are all tight. These examples are:

  (i) The 9-vertex triangulation $\mathbb{C}P_9^2$ of $\mathbb{C}P^2$ due to Kühnel (in [26]),

  (ii) the 16-vertex triangulation of a $K3$-surface due to Casella and Kühnel (in [11]),

  (iii) two 13-vertex triangulations of $S^3 \times S^3$ due to Lutz (in [29]), and

  (iv) six 15-vertex triangulations of homology $\mathbb{H}P^2$ (three due to Brehm and Kühnel in [9] and three due to Lutz in [30]).

(f) Apart from the above list, we know only two tight triangulated manifolds. These are:

  (i) A 15-vertex triangulation of $(S^3 \ltimes S^1)\#(\mathbb{C}P^2)^{\#5}$ due to Lutz (in [29]). It is 2-neighbourly, non-orientable, $\mathbb{Z}_2$-tight and in $\mathcal{W}_2(4)$.

  (ii) A 13-vertex triangulation of $SU(3)/SO(3)$ due to Lutz (in [29]). It is 3-neighbourly, orientable, $\mathbb{Z}_2$-tight and in $\mathcal{W}_3(5)$.

The tightness of these two examples do not follow from the results presented here.

Corollary 5.11 implies that all the triangulations in Example 6.11 (a), (b) and (c) are strongly minimal. By Theorem 4.4 of [35], all the triangulations in Example 6.11 (d) and (e) are minimal. As far as we know, the minimality of the triangulations in Example 6.11 (f) is an open problem.

Finally, we have the question of how to get new triangulations meeting the hypothesis of Propositions 5.8 and 5.14. In particular, we may ask:

Question 6.12. Is there a 20-vertex triangulation of $(S^2 \times S^1)^{\#12}$ or $(S^2 \mathbin{\underline{\times}} S^1)^{\#12}$ in $\mathcal{W}_1^*(3)$ or a 20-vertex triangulation of $(S^3 \times S^2)^{\#13}$ in $\mathcal{W}_2^*(5)$?

We might also ask if there are examples of triangulations from $\mathcal{W}_k(d) \backslash \mathcal{W}_{k-1}(d)$ satisfying the hypothesis of Proposition 5.15 (cf. Remark 5.16). We suspect that there may not exist any such examples!

We do not know for a fact that, for $1 < l < (d-1)/2$, the members of $\mathcal{K}_l(d)$ (or even of $\mathcal{W}_l(d)$) actually attain equality in Conjecture 6.7. Thus, all parts of this conjecture are wide open for $l > 1$. Notice that, as a consequence of Proposition 5.5, the members of $\mathcal{W}_l^*(d)$ do attain equality in Conjecture 6.8 for $1 \leq l < (d-1)/2$. However, we do not know if, more generally, the members of $\mathcal{K}_l^*(d)$ attain these equalities for $1 \leq l < (d-1)/2$, as conjectured. The case $l = 1$ is unrevealing since in this case $\mathcal{W}_1(d) = \mathcal{K}_1(d)$ and $\mathcal{W}_1^*(d) = \mathcal{K}_1^*(d)$ (Corollary 2.16). Thus, the most important question raised by this paper is whether (and to what extent) the results can be extended from k-stellated spheres to k-stacked spheres. A good place to begin this investigation is to address the following:

Question 6.13. Is Proposition 5.3 (on the sigma-vector of k-stellated spheres) valid for k-stacked spheres?

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